12.307 Project I: Radial Inflow

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Abstract

A rotating bucket with a hole in the center of its bottom is used to model a low-pressure weather system in a rotating frame. As the water drains, it rotates at a rate faster than that of the frame as it approaches the center in order to conserve angular momentum. As the fluid spins, the Coriolis force and centrifugal force balance out the pressure gradient to varying degrees. To what extent each force acts against the pressure gradient is captured in the non-dimensional number called the Rossby number. The Rossby number decreases with the size of the radius.

1 Introduction

The purpose of this experiment is to demonstrate the effect of the rotation of fluids in a rotating system under the influence of a pressure gradient, with particular focus on the dependence of the Rossby number on radius from the axis of rotation. When a pressure gradient is present, a fluid will move radially toward places of low pressure according to the pressure gradient. However, in a rotating frame the fluid will swirl around the source of pressure and will not just travel radially. This effect is due to the presence of the Coriolis force and the centrifugal force, which both work to balance the pressure gradient.

The Earth's weather exists in a rotating frame. The Earth rotates at a near constant rotational velocity and has a period of approximately 24 hours. The effect of this rotation is the creation of what seems to be an "inertial" frame on the Earth in which stationary objects on the surface remain stationary in that frame (i.e. they have the same angular velocity). The motion of weather systems can be primarily attributed to two factors: the presence of a pressure gradient and conservation of momentum.

The pressure gradient arises as a result of a temperature gradient from the equator to the poles. The poles receive significantly less direct sunlight than the equator and radiate more heat,

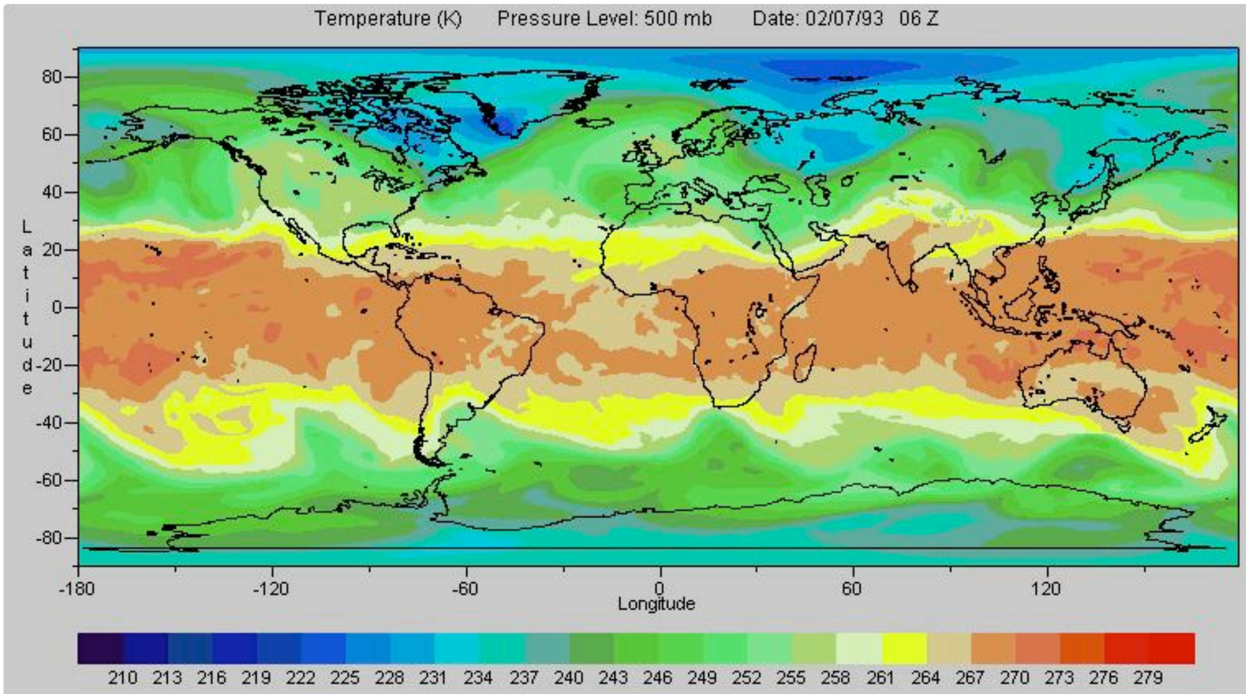

Figure 1: Temperature gradient at 500 mb isobaric surface at 06 Z on 02/07/93.

making them colder than the equator. Cold air is denser than warm air by the ideal gas law, which states that PV=nRT; pressure times the volume equals the number of molecules times the ideal gas constant time the temperature. So, for the same volume of air, a lower temperature implies a lower pressure. This means that given two columns of air with a 1 square meter footprint and the same, that more cold air would be fit into the column than warm air, or that for the same amount of air in a column, the column with the warmer air would have to be far taller. The pressure at a particular altitude is determined by the amount of air above that position, thus, at higher altitudes, since the column of cold air is significantly shorter, the pressure is less. This implies that the pressure at a particular altitude is on average less at the poles, creating a pressure gradient that increases as you move away from the equator.

Due to the pressure gradient, air parcels at the equator begin to be forced toward the poles. Since every parcel in the air was initially rotating, albeit with the same period as the Earth, they have a certain angular momentum. As the parcels of air move north, their distance from the axis, i.e. the radius of their rotation, becomes progressively smaller. Due to the law of conservation of momentum, these parcels must conserve momentum; to achieve this, the air parcels rotate around the earth at faster and faster speeds.

The rotation of individual weather systems is due to the balance between the Coriolis and centrifugal forces and the pressure gradient. Both the Coriolis and centrifugal forces are what are sometimes called "pseudo-forces" because their effects are only noticeable while in a rotating frame. Nevertheless, they are of paramount importance when considering weather systems on the earth. The Coriolis force describes the cause of the apparent deflection of objects in the rotating frame when they are moving straight in the absolute frame. The position of the object relative to the rotating frame will not only change according to the motion in the absolute frame of the object, but also the rotation of the rotating frame. One way to visualize this phenomenon is to accelerate a ball straight on a rotating carousel and throwing a ball. From your reference point, although the ball was accelerated directly away from the source, it does not appear to fly straight. Instead, it curves in the direction opposite the rotation of the carousel. Much the same way, the Earth's rotation apparently deflects air motion perpendicularly to the rotational axis.

The centrifugal force explains the apparent outward acceleration of an object in rotation. To understand the centrifugal force better, consider a ball attached to a rope swinging in a circle. As the ball rotates, it is pulled inwards due to centripetal force as it attempts to move tangentially due to inertia, but in the frame of the ball, it feels as if it is constantly being pulled away from the center.

2 Experiment

2.1 Setup

A low-pressure system was created by taking a bucket with a whole in the center of its bottom. The bucket was filled with water, which was allowed to drain when a plug was removed from the hole. In order to simulate the rotation of the earth, the system was placed in a tank rotating at a constant angular velocity.

2.2 Theory

In low-pressure systems, both the Coriolis force and the centrifugal force work to counteract the pressure gradient. The relationship that describes this balance is called the gradient wind balance. In order to get this relationship, let us begin with the hydrostatic balance of the system, which states that the pressure at a certain height, call it z, must be able to support the fluid above it. This statement is written

$$p = \rho g(H - z), \tag{1}$$

where p is pressure, ρ is the density of the fluid, g is acceleration due to gravity, H, which is dependent on radius, is the height of the surface of the fluid, and z is a given height from the bottom of the tank. While H and z vary, ρ and p are constant quantities.

For azimuthal velocities where the azimuthal velocity and the velocity in the radial direction are of approximately equal magnitudes, the effects of the centrifugal force are the most important. This is primarily true for smaller radii where the parcels of fluid are rotating in tight spirals. At this point, the pressure gradient is balanced by the centrifugal force,

$$\frac{V_\theta^2}{r} = \frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{2}$$

where V_θ is the azimuthal velocity in the absolute frame. Using equation (1), equation (2) can be rewritten as

$$\frac{V_\theta^2}{r} = g\frac{\partial H}{\partial r}. \tag{3}$$

Now, let us put the expression in terms of velocities relative to the rotating frame instead of the absolute frame. Given the velocity relative to the rotating frame is v_θ, the velocity in the absolute frame is simply that velocity plus the velocity due to the rotation of the frame

$$V_\theta = v_\theta + \Omega r, \tag{4}$$

Ω being the rate of rotation of the frame. Thus,

$$\frac{V_\theta^2}{r} = \frac{(v_\theta + \Omega r)^2}{r} = \frac{v_\theta^2}{r} + 2\Omega v_\theta + \Omega^2 r \tag{5}$$

Therefore,

$$\frac{v_\theta^2}{r} + 2\Omega v_\theta + \Omega^2 r = g\frac{\partial H}{\partial r}. \tag{6}$$

Simplifying by integration defining the a quantity h

$$h = H - \frac{\Omega^2 r^2}{2g}. \tag{7}$$

So, substituting h and then rearranging,

$$\frac{v_\theta^2}{r} + 2\Omega v_\theta = g\frac{\partial h}{\partial r}. \tag{8}$$

This equation is known as the gradient wind balance. The $\frac{v_\theta^2}{r}$ term is, as explained previously, due to the centrifugal force while $g\frac{\partial h}{\partial r}$ is the pressure gradient and $-2\Omega v_\theta$ is known as the Coriolis acceleration. For small radii and higher velocities, the centrifugal force is more important in balancing the pressure gradient, but for larger radii and lower velocities the Coriolis term becomes more important. In order to have a way to tell when which force becomes dominant, let us introduce a new term that is the ratio between the two to show their relative strength:

$$R_0 = \frac{|v_\theta|}{2\Omega r} \tag{9}$$

This non-dimensional term is known as the Rossby number. When R_0 is less than 1, the Coriolis force is dominant while for R_0 greater than 1 the centrifugal force is dominant. So, for R_0<<1, the expression is approximately

$$2\Omega v_\theta = g\frac{\partial h}{\partial r} \quad : \quad \text{geostrophic balance} \tag{10}$$

Alternatively, if R>>1,

$$\frac{v_\theta^2}{r} \approx g\frac{\partial h}{\partial r} \quad : \text{ cyclostrophic balance.} \qquad (11)$$

2.3 Results

To model a fluid in a pressure gradient, water was put into a bucket and the allowed to be spun into solid body rotation. After reaching a state of equilibrium, a plug at the bottom of the tank was pulled and the water allowed to drain, creating in effect a low pressure system centered at the hole. The bucket was spun at three different rotational velocities: 10, 5, and 15 rpm. Paper dots were placed in the water and their position was tracked with a particle tracker. With higher velocities, the particles would go around more times.

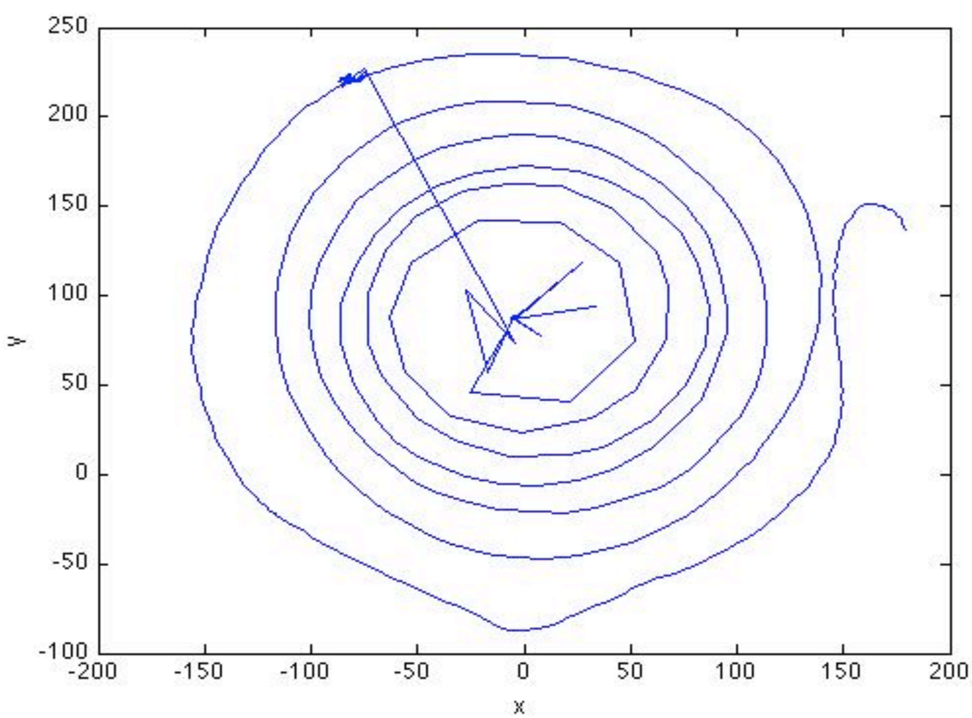

Figure 2: Track of particle in the bucket rotating at 5 rpm. Measurements are in pixels.

As the water drained, the particles near the inside had a very high velocity and made a full period many times in the time the bucket took to go around once. However, near the edge of the bucket, the particles moved very slowly or almost not at all. The dominant force at a particular radius can be found by finding the Rossby number at that radius. To do so, data points were collected using the particle tracker. A center was determined and using that center, radial particle positions were calculated using the Pythagorean Theorem

$$r = \sqrt{x^2 + y^2}\,. \tag{12}$$

The angle to the point was then calculated by taking the arctangent of the x and y coordinates. With this information, the Rossby number could be computed for a particular position and hence a particular radius. The formula used to calculate the Rossby number was derived by using the fact that v_θ can be represented as the change in angle over the change in time. The resulting formula is

$$R_0 = \frac{\theta_{i+1} - \theta_i}{2\Omega(t_{i+1} - t_i)}\,. \tag{13}$$

Using this formula, Rossby number near the hole are indeed found to be greater than those near the radius of the bucket. This confirms expectations. A graph plotted using MATLAB showing the Rossby number against the radius is found below. The experimental results are compared with a theoretical result.

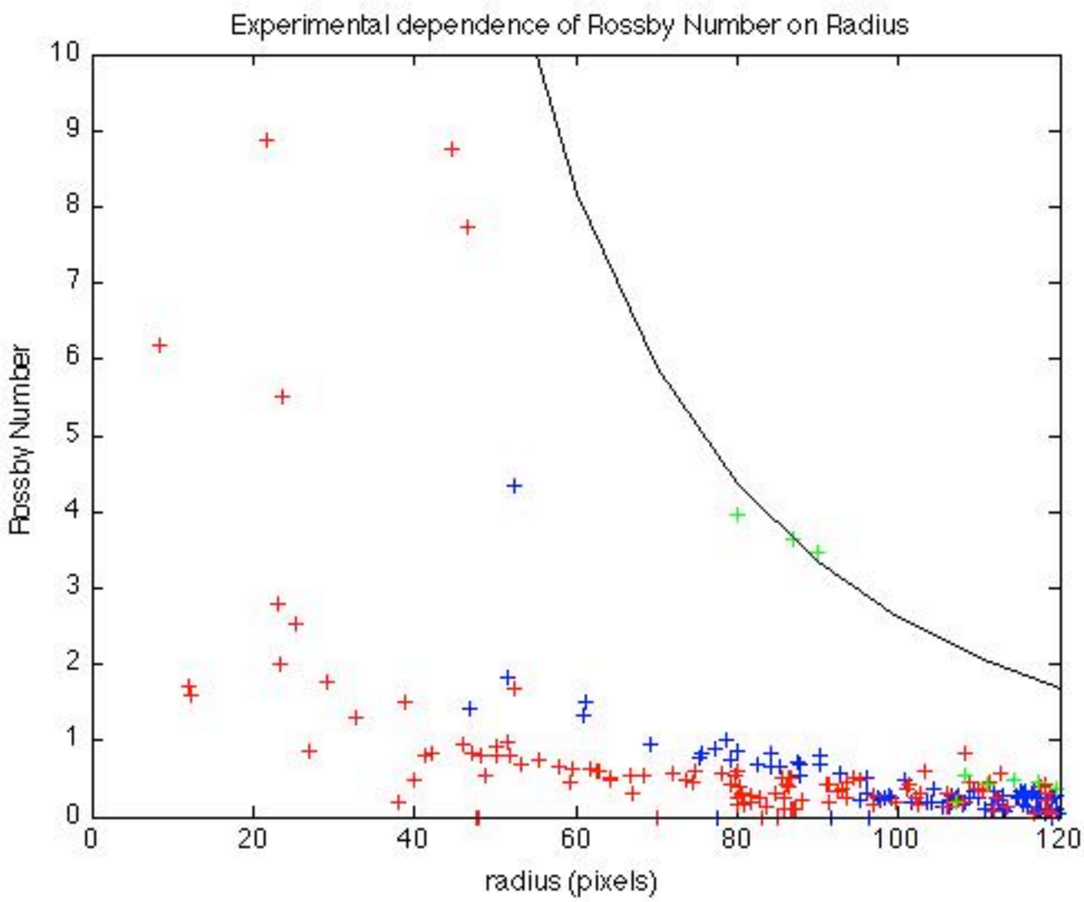

Figure 3: The Rossby Number against radius. The blue points are at 10 rpm, the red at 5 rpm, and the green at 15 rpm. The black curve is the theoretical curve, using 250 pixels as the approximate radius of the bucket.

The theoretical value for the Rossby number as indicated by the solid line is found by using an approximation of the actual Rossby number that relies on the fact that the fluid conserves angular momentum. Given that the azimuthal speed V_θ of a particle in the tank with referene to the absolute frame is given by

$$V_\theta = v_\theta + \Omega r \quad (14)$$

where v_θ is the azimuthal speed with respect to the tank and Ωr is the speed of the tank at a particular radius. Angular momentum is represented by the expression

$$V_\theta r = const. = \Omega r_1^2 \quad (15)$$

where r_1 is the radius of the bucket. By combining equations (14) and (15), the expression

$$v_\theta = \Omega \frac{r_1^2 - r^2}{r} \quad (16)$$

is obtained, which can be inserted in equation (9) to give the approximation

$$R_0 = \frac{1}{2}\left(\left(\frac{r_1}{r}\right)^2 - 1\right), \quad (17)$$

where r_1 is the radius of the low pressure system and r is a variable radius, which holds assuming momentum is conserved.

As can be seen, the data points follow the general trend of the theoretical curve, however they do not follow it exactly. The fact that the data points do not follow the theoretical curve more closely can be explained by the choice of r_1. The radius of the system was chosen to be the radius of the bucket, although it is difficult to determine if this was exactly the case. It is possible that the radius of the low pressure system was smaller than the radius of the bucket, which would explain while the expected value of the Rossby number is generally larger than what is observed. That is, near the radius of the bucket the particles were hardly moving. Nevertheless, the data do seem to indicate that the expectation that the Rossby number should decrease as radius increases is correct.

3 Real-World Comparisons

Experimental results are only useful in their ability to act as an analog to real-world situations. The relevance of the experimental results is in comparison to the flow of air in low-pressure regions on the Earth. Thus, the purpose of this section will be to draw comparisons between the experimental results and an actual storm. The storm data used were collected from hurricane Jeanne on September 22, 2009 as it approached the east coast of Florida.

Hurricanes are low-pressure systems, like the one created in the tank experiment described above, so similar results and a similar theory are expected. The same way the water in the bucket swirled around the bucket toward the drain at the center, the air in a low-pressure system swirls around the source of low pressure. This phenomenon can clearly be seen in the diagram below; the air rotates around, flowing inward toward the source of the system.

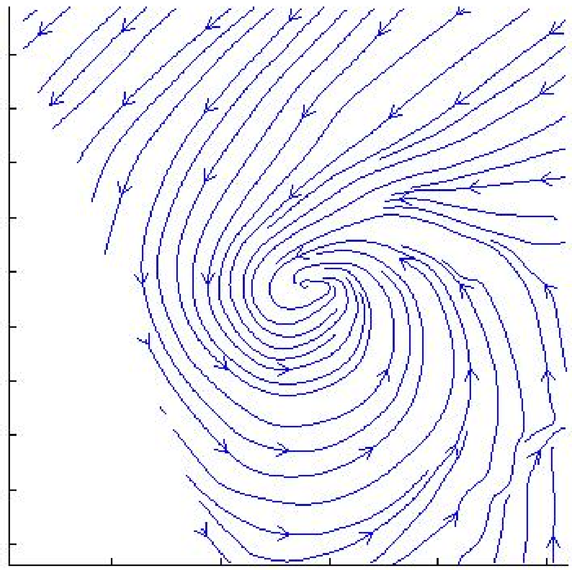

Figure 4: The direction of airflow in Hurricane Jeanne. 9/22/2004

The system behaves much the same way as the tank experiment. A direct analogy can be drawn between equation (8) and what is observed in the hurricane. By modifying equation (8),

$$\frac{v_\theta^2}{r} + f v_\theta = g\frac{\partial h}{\partial r}, f = 2\Omega \sin\varphi \quad . \tag{18}$$

It is important to notice that instead of 2Ω, on the earth the equation includes $2\Omega\sin\varphi$, where φ is in degrees of latitude. So, the system will be in cyclostrophic balance when v_θ^2 balances the pressure gradient, and geostrophic balance when the fv_θ term balances the pressure gradient. Cyclostrophic balance occurs at small radii when the velocity is high, as when the particles approached the drain in the tank experiment. Geostrophic balance occurs at larger radii and higher velocities.

MATLAB was used to plot contours of the wind speed of the storm, which can be seen below.

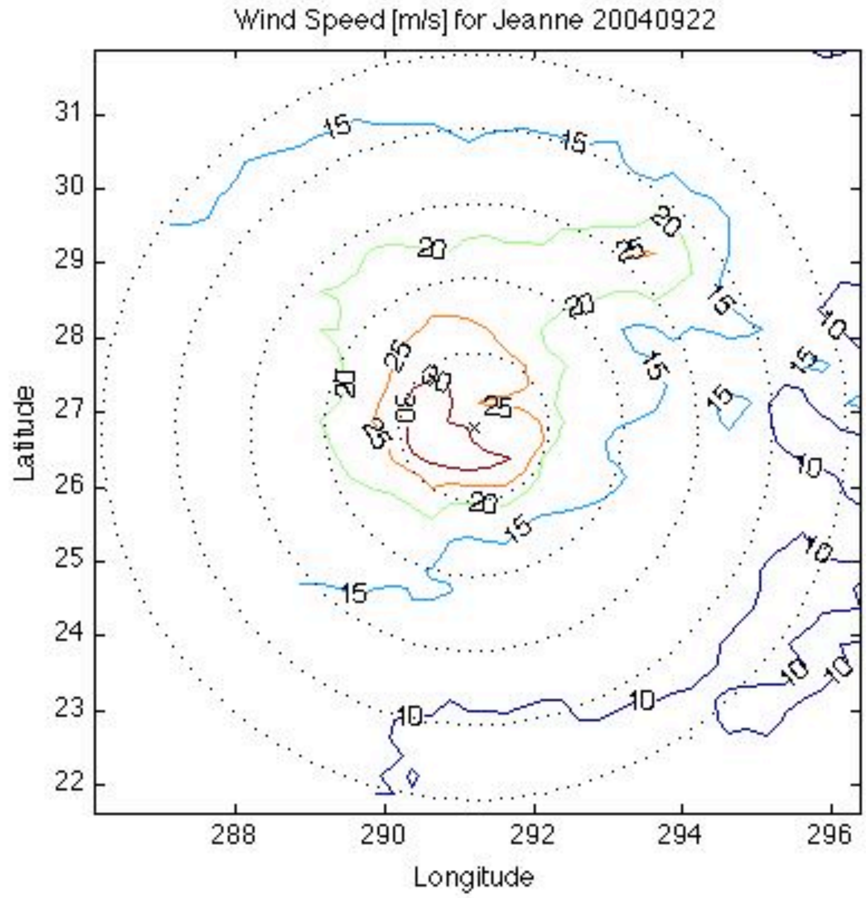

Figure 5: Contour plot of wind speed in hurricane Jeanne. 9/22/2004

The same way the equation for the balance of forces needs to be altered to generalize to the earth, the equation for the Rossby number (9) also needs to be changed. The new equation for the Rossby number is

$$R_0 = \frac{|v_\theta|}{2\Omega \sin \varphi r}. \qquad (19)$$

Using the wind speeds along increasing radii to the north and south of the eye of the storm, the Rossby number was plotted for various radii using equation (19). A theoretical curve was fitted to this graph by taking the radius of the storm to be about 12 meters per second, or about 200,000 meters from the eye. This value for the radius of the storm was chosen after testing different values for the radius in the calculation of the Rossby number and picking the one that most closely fit the data points. At this radius of the storm, wind speeds were small enough that it is feasible that they are not caused as a direct result of the low-pressure system. The wind speeds along the north-south diameter along with the fitted theoretical curve can be seen below.

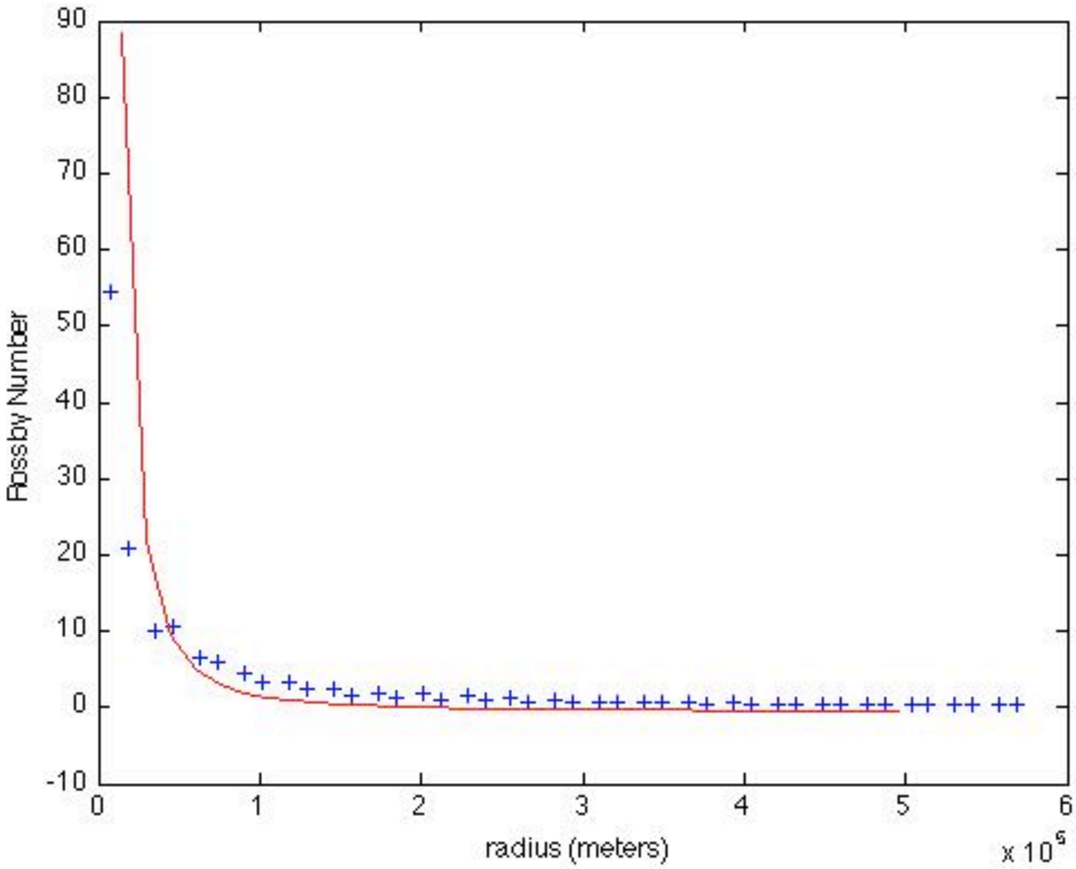

Figure 6: The Rossby number against radius measured in meters.

While the storm better demonstrates the dependence of the Rossby number on radius than the experiment, it still confirms the implications drawn from the experimental results. From the graph above, it is clear that as larger radii in the storm are considered, the Rossby number decreases according to approximately an inverse square function.

Hurricane Jeanne obeys the gradient wind equation (8), so we can use our results for the Rossby number to draw conclusions about the force balance. Looking at figure 5, it is clear that as smaller radii, $R_0 >> 0$. Likewise, is evident that at large radii, R_0 is very small. The point at which both forces about equally balance out the pressure gradient, i.e, $R_0 \approx 1$, is when the radius is about 2.25 degrees of latitude, or 247,500 meters from the center of the storm.

4 Conclusion

The Rossby number is an efficient way of determining which force is predominant at a particular radius in a rotating system. Using the Rossby number in combination with the gradient wind balance equation gives a relatively accurate description of the force balance at that point. When the Rossby number is very small, the Coriolis for is dominant and the system is in cyclostrophic balance, but when the Rossby number is very large the centrifugal force is dominant and the system is in geostrophic balance. Large Rossby numbers are found at small radii where velocities are high and the centrifugal force balances the pressure gradient. Small Rossby numbers are found at large radii where velocities are small. An accurate analogy to a low-pressure system in the atmosphere is water draining out of a bucket with a hole in its bottom in a rotating tank.

Reference

[1] Marshall, John. "12.307 Project 1: Radial Inflow Experiment"

[2] *http://www.swa.com/ALD/LidarProducts/npoess/T213stats.htm*

(I'm still not sure how to cite source [1] when almost all of my theoretical calculations came from there)