Radar Coding and Signal Processing

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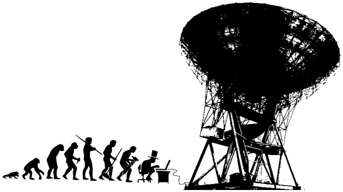
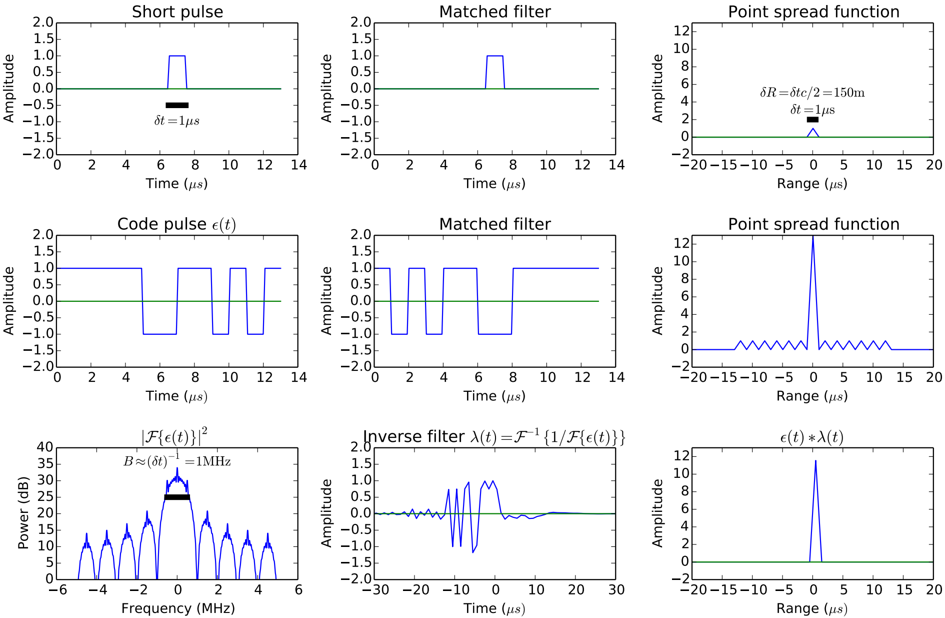


Working with high power large aperture radars, you get a chance to learn (Josh Semeter):

- Geophysics, plasma physics, atmospheric science, and climate research...
- Electronics: high power amplifiers, RF engineering, digital signal processing, radar transmit coding...
- Computer science, dealing with lots of data, writing software...
- **Signal Processing: Bayesian Statistics, inverse problems, applied mathematics, radar transmit coding**
- Astronomy

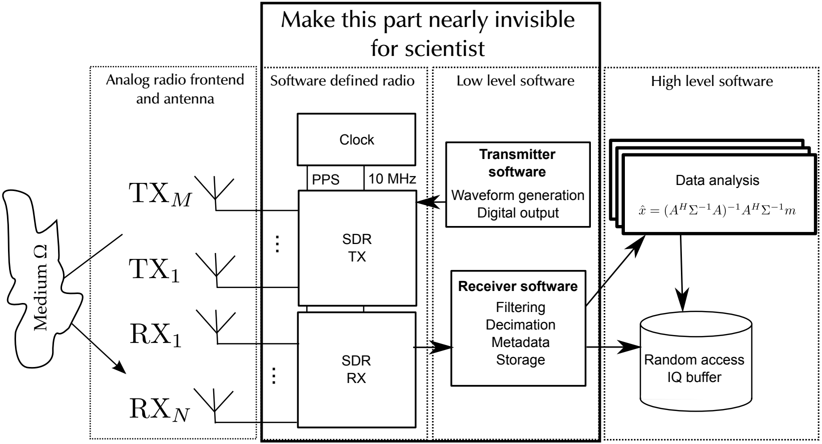
Range resolution, bandwidth, compressed power, range



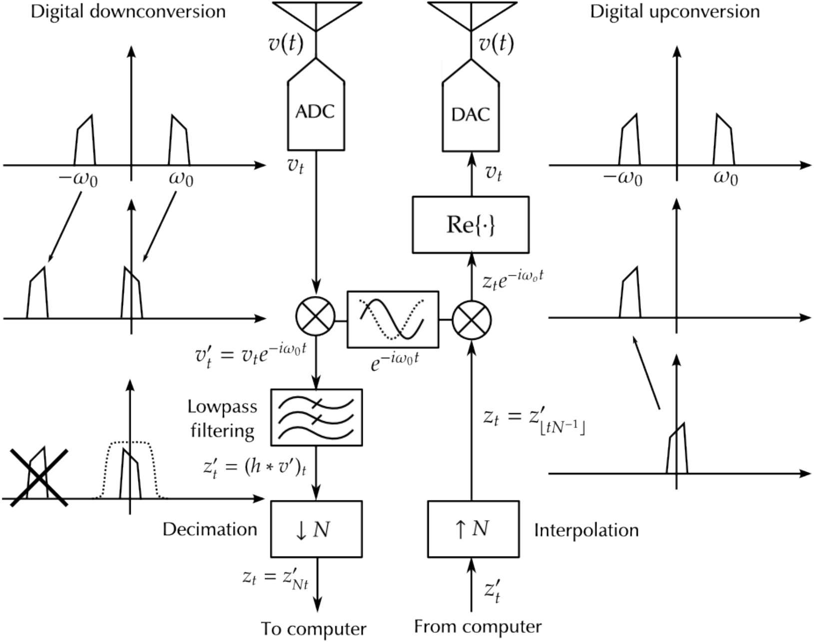
Why bother?

- Radar experiment design determines: range resolution, range extent, spectral resolution, spectral extent, and *error bars*.

Radar architecture (Sofrware Defined Radio)



What is a baseband signal?



What is a baseband signal?

The Fourier transform of a real valued signal $x(t) \in \mathbb{R}$ is conjugate symmetric $x(\omega) = x(-\omega)^*$, which directly follows from the definition:

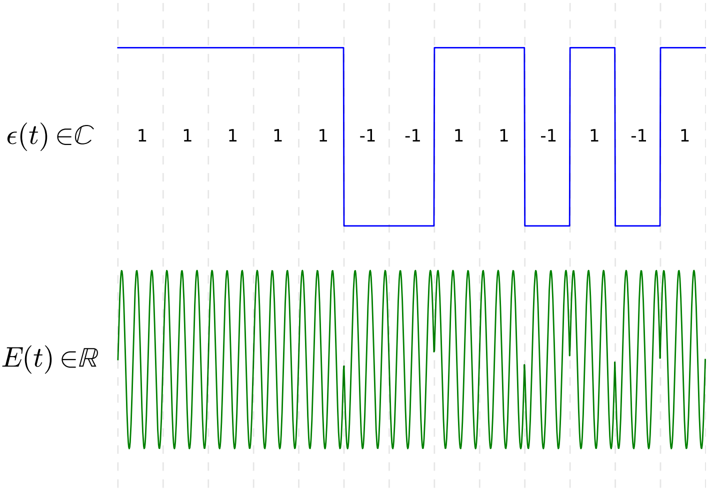
$$x(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt \tag{1}$$

Band limited signal only contains information in a band B around ω_0, so it is natural to consider only the portion of the spectrum that contains the information:

$$z(t) = e^{-i\omega_0 t} \int_{\omega_0 - B/2}^{\omega_0 + B/2} x(\omega) e^{i\omega t} d\omega \tag{2}$$

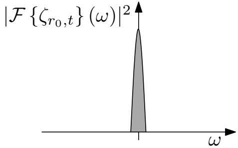
The Nyquist-Shannon theorem also says that a sample rate of B is sufficient to retain all information within the band of interest, so we end up with a discretized signal $z_n = z(n\Delta t) \in \mathbb{C}$.

Radar waveform: complex baseband vs. real signal

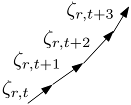


Coherent vs. Incoherent

Coherent target

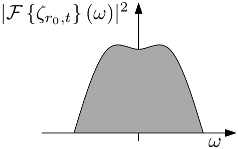


$m_t = \sum_r \epsilon_{t-r}\zeta_{r,t} + \xi_t \approx \sum_r \epsilon_{t-r}\zeta_r + \xi_t$

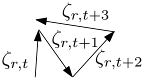


m_t →
- Matched filter
- Inverse filter
- Matrix deconvolution
- Pulse-to-pulse Doppler

Incoherent target



$m_t = \sum_r \epsilon_{t-r}\zeta_{r,t} + \xi_t$



$\mathrm{E}\left\{m_t\overline{m_{t+\tau}}\right\}$ →
- Lag-profile inversion
- Full-profile inversion

Summary

- Radar waveform, baseband signal
- Probability theory basics – when dealing with mostly noise dominated signals, we need to use statistics
- Functional analysis aspects – mostly Fourier domain, convolution, and circulant matrices $\rightarrow$ efficient numerical solutions and easier to study analytically
- Coherent target and Incoherent target
- Bandwidth $=$ Resolution
- Time domain. Integrated power $=$ Peak power
- Typical codes: pseudorandom codes, alternating codes, optimized codes, Barker codes, complementary codes, long pulse, amplitude modulated codes.

Measurement model

Measurement model or *forward model* relates measurements to a model that describes the measurements with the help of parameters $\mathbf{x}$ and random variables $\boldsymbol{\xi}$. In the case of radar measurement, nearly all models are linear:

$$\mathbf{m} = \mathbf{A}\mathbf{x} + \boldsymbol{\xi} \tag{3}$$

Here $\mathbf{A}$ is a function that can "simulate" the measurements, given the correct values of parameters $\mathbf{x}$.
$\mathbf{m} \in \mathbb{C}^{M \times 1}$, measurement vector
$\mathbf{A} \in \mathbb{C}^{M \times N}$, theory matrix
$\mathbf{x} \in \mathbb{C}^{N \times 1}$, unknown parameters
$\boldsymbol{\xi} \in \mathbb{C}^{M \times 1} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \boldsymbol{\Sigma})$, measurement noise.

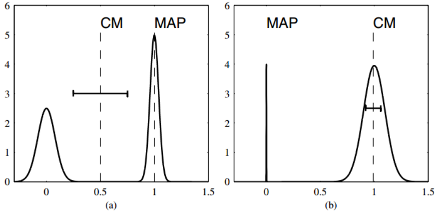
Statistical Inverse Problem, Bayes' Theorem

$$p(\mathbf{x}|\mathbf{m}) = \frac{p(\mathbf{m}|\mathbf{x})p(\mathbf{x})}{p(\mathbf{m})}$$

$$p(\mathbf{x}|\mathbf{m}) \propto p(\mathbf{m}|\mathbf{x})p(\mathbf{x})$$

Where
$p(\mathbf{m})$ is probability of measurements
$p(\mathbf{x})$ is *prior* distribution of unknown $\mathbf{x}$
$p(\mathbf{m}|\mathbf{x})$ is the *likelihood* function
$p(\mathbf{x}|\mathbf{m})$ is the *a posteriori* distribution

Estimators



(Kaipio & Somersalo 2004)

Maximum a posteriori:

$$x_{\mathrm{MAP}} = \underset{\mathbf{x}}{\operatorname{argmax}}\, p(\mathbf{x}|\mathbf{m}) \tag{4}$$

Conditional mean:

$$x_{\mathrm{CM}} = \int \mathbf{x} p(\mathbf{x}|\mathbf{m}) d\mathbf{x} \tag{5}$$

Maximum likelihood:

$$x_{\mathrm{ML}} = \underset{\mathbf{x}}{\operatorname{argmax}}\, p(\mathbf{m}|\mathbf{x}) \tag{6}$$

Linear Model, Complex Valued Problems

If noise is *proper complex* Gaussian noise, ie., $\mathrm{E}\boldsymbol{\xi}\boldsymbol{\xi}^{\mathsf{T}} = \mathbf{0}$ and $\mathrm{E}\boldsymbol{\xi}\boldsymbol{\xi}^{\mathsf{H}} = \boldsymbol{\Sigma}$, then:

$$p(\mathbf{m}|\mathbf{x}) = \frac{1}{\pi^N |\boldsymbol{\Sigma}_{\mathrm{post}}|} \exp\left\{ -(\mathbf{m} - \mathbf{A}\mathbf{x}_{\mathrm{ML}})^{\mathsf{H}} \boldsymbol{\Sigma}_{\mathrm{post}}^{-\mathbf{1}} (\mathbf{m} - \mathbf{A}\mathbf{x}_{\mathrm{ML}}), \right\} \tag{7}$$

with

$$\boldsymbol{\Sigma}_{\mathrm{post}} = (\mathbf{A}^{\mathsf{H}} \boldsymbol{\Sigma}^{-\mathbf{1}} + \boldsymbol{\Sigma}_{\mathbf{prior}})^{-\mathbf{1}} \tag{8}$$

and

$$\mathbf{x}_{\mathrm{ML}} = \boldsymbol{\Sigma}_{\mathrm{post}} \mathbf{A}^{\mathsf{H}} \boldsymbol{\Sigma}^{-\mathbf{1}} \mathbf{m} \tag{9}$$

Statistical Inverse Problem, Maximum a Posteriori, Regularization

If we additionally make the prior assumption that our unknown is a Gaussian normal random variable distributed as $\mathbf{x} \sim N(\mathbf{0}, \Sigma_{\text{prior}})$, then

$$p(\mathbf{x}|\mathbf{m}) = \frac{1}{\sqrt{2\pi^N |\boldsymbol{\Sigma}_{\text{post}}|}} \exp\left\{ -\frac{1}{2}(\mathbf{m} - \mathbf{A}\mathbf{x}_{\text{MAP}})^{\mathsf{T}} \boldsymbol{\Sigma}_{\text{post}}^{-\mathbf{1}} (\mathbf{m} - \mathbf{A}\mathbf{x}_{\text{MAP}}) \right\}$$

Posteriori covariance

$$\boldsymbol{\Sigma}_{\text{post}} = (\mathbf{A}^{\mathsf{T}} \boldsymbol{\Sigma}^{-\mathbf{1}} \mathbf{A} + \Sigma_{\text{prior}}^{-\mathbf{1}})^{-\mathbf{1}} \tag{10}$$

Maximum likelihood estimator:

$$\mathbf{x}_{\text{MAP}} = \mathbf{x}_{\text{CM}} = \boldsymbol{\Sigma}_{\text{post}} \mathbf{A}^{\mathsf{T}} \boldsymbol{\Sigma}^{-\mathbf{1}} \mathbf{m} \tag{11}$$

Linear Model, Complex Valued Problems

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with

$$\boldsymbol{\Sigma}_{\text{post}} = (\mathbf{A}^{\mathsf{H}} \boldsymbol{\Sigma}^{-\mathbf{1}} \mathbf{A})^{-\mathbf{1}} \tag{13}$$

and

$$\mathbf{x}_{\text{ML}} = \boldsymbol{\Sigma}_{\text{post}} \mathbf{A}^{\mathsf{H}} \boldsymbol{\Sigma}^{-\mathbf{1}} \mathbf{m} \tag{14}$$

Convolution equation, circulant matrix

The radar pulse convolved with a stationary target is a convolution equation:

$$m_t = \sum_{r=0}^{R} \epsilon_{t-r}\zeta_r + \xi_t \tag{15}$$

Can be expressed as a matrix equation

$$\mathbf{m} = \mathbf{A}\mathbf{x} + \boldsymbol{\xi} \tag{16}$$

$$\begin{bmatrix} m_0 \\ m_1 \\ \vdots \\ m_N \end{bmatrix} = \begin{bmatrix} c_0 & c_{n-1} & \dots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \dots & c_1 & c_0 \end{bmatrix} \begin{bmatrix} \zeta_0 \\ \zeta_1 \\ \vdots \\ \zeta_R \end{bmatrix} \tag{17}$$

This type of an operator is ubiquitous in radar measurement theory.

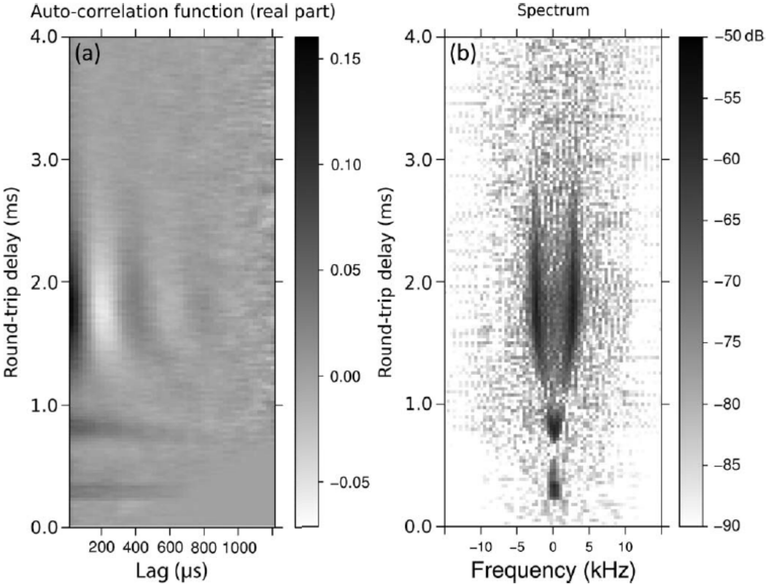
Correlation Estimate

Calculating the posteriori covariance is sometimes very expensive. If we can formulate the problem in such a way that $\boldsymbol{\Sigma} \approx \beta \mathbf{I}$ $(\mathbf{A}^{\mathsf{H}}\mathbf{A} + \boldsymbol{\Sigma}_{\text{prior}^{-1}})^{-\mathbf{1}} \approx \alpha \mathbf{I}$ or even $(\mathbf{A}^{\mathsf{H}}\mathbf{A})^{-\mathbf{1}} = \alpha \mathbf{I}$, then we can simplify our MAP and ML estimates.

$$\mathbf{x}_{\mathsf{MAP}} = \mathbf{A}^{\mathsf{H}}\mathbf{m} \tag{18}$$

Can be seen as a form of regularization to achieve computational speed at the expense if a slightly "wrong" prior assumption.
Examples: plasma line, match function in radar, discrete Fourier transform, Lomb-Scargle periodogram, Alternating codes, pseudorandom codes, complementary codes, Barker codes...
(Sulzer 1986; Lehtinen and Häggström 1987)

Fourier relationship of an ACF and Spectrum (1/2)



Fourier relationship of an ACF and Spectrum (2/2)

An autocorrelation function $\sigma(\tau) = \mathrm{E}z(t+\tau)\overline{z(t)}$ can be expressed as the convolution of a signal $z(t)$ with itself:

$$\sigma(\tau) = \lim_{t\to\infty} \frac{1}{T} \int_0^T z(t+\tau)\overline{z(t)}dt \tag{19}$$

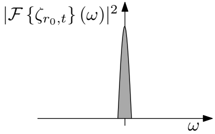
The above equation gives us a hint that the autocorrelation function is related with power spectral density (a convolution is multiplication is frequency domain). The Weiner-Khinchin theorem:

$$S(\omega) = \int \sigma(\tau) e^{-i\omega\tau} d\tau \tag{20}$$

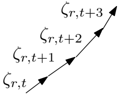
$$\sigma(\tau) = \int S(\omega) e^{i\omega\tau} d\omega \tag{21}$$

Coherent vs. Incoherent

Coherent target

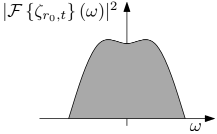


$m_t = \sum_r \epsilon_{t-r}\zeta_{r,t} + \xi_t \approx \sum_r \epsilon_{t-r}\zeta_r + \xi_t$

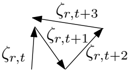


$m_t \rightarrow$
- Matched filter
- Inverse filter
- Matrix deconvolution
- Pulse-to-pulse Doppler

Incoherent target

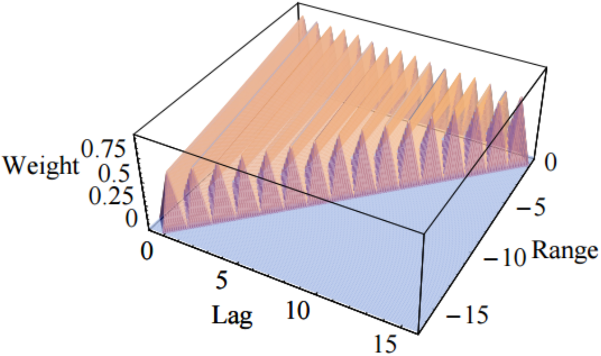


$m_t = \sum_r \epsilon_{t-r}\zeta_{r,t} + \xi_t$



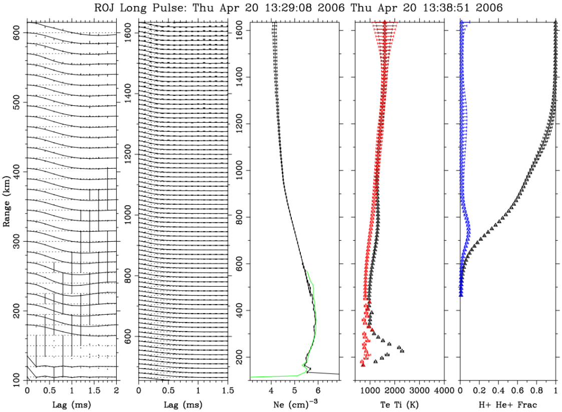
$\mathrm{E}\{m_t\overline{m_{t+\tau}}\} \rightarrow$
- Lag-profile inversion
- Full-profile inversion

Full-profile inversion



For long-pulse, the ambiguity is not invertible, i.e., $\mathbf{A}^H\mathbf{A}$ is singular. In this case, one can still proceed to fit plasma parameters on the point-spread function, or ambiguity function. The solution is solvable by making a prior assumption of continuity of plasma parameters.

Full-profile inversion



- Pros: very sensitive due to largest possible scattering volume and smallest possible receiver bandwidth.
- Cons: range extended correlations

(Hysell et.al., 2008; et.al., Holt 1992)

Lag-profile inversion

Discretized incoherent scatter radar equation:

$$m_t = \sum_r \epsilon_{t-r}\zeta_{r,t} + \xi_t \qquad (22)$$

Here m_t is measured baseband signal, ϵ_{t-r} is the transmit waveform, ξ_t is receiver noise, and $\zeta_{r,t}$ is the incoherent backscatter process, the collective return from the bazillions of scatterers moving in various directions within the radar volume defined propagation time r and t.
In radar we often use propagation time r from transmitter to target and target to receiver (in the monostatic case, round-trip time $r = 2R/c$).

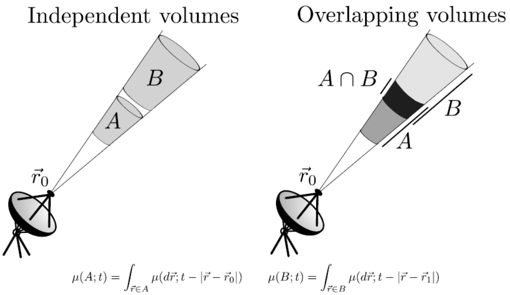
Lag-profile inversion

If the change of $\zeta_{r,t}$ is slow enough that $\zeta_{r,t} \approx \zeta_r$, then the equation reduces to a convolution equation. This is true in the D-region with most radars, and also with perpendicular to the magnetic field measurements at Jicamarca.

$$m_t = \sum_r \epsilon_{t-r}\zeta_r + \xi_t \tag{23}$$

However, in the E and F-regions, $\zeta_{r,t}$ already changes significantly during the time that the pulse travels through the scattering volume. How do we solve the problem then?

Lag-profile inversion



We assume that the second order products (the ACF) $\mathrm{E}\, \zeta_{r,t}\overline{\zeta_{r,t+\tau}} = \sigma_r^\tau$ are unchanged over a certain observation period. This is the *wide sense stationary uncorrelated scattering medium* assumption (Clarke 1968, van Trees 2002).
To estimate σ_r^τ, we can look at lagged products. Dropping all the zero mean terms, we get:

$$m_t\overline{m_{t+\tau}} = \sum_r \epsilon_{t-r}\overline{\epsilon_{t-r+\tau}}\sigma_r^\tau + \xi_t'. \tag{24}$$

Lag-profile inversion

In linear equation form

$$\mathbf{m}^{\boldsymbol{\tau}} = \mathbf{W}^{\tau}\boldsymbol{\sigma}^{\tau} + \boldsymbol{\xi}'. \quad (25)$$

The noise term ξ_t' is not Gaussian white noise, and also contains self-noise, but it approaches the Gaussian distrubution when we average lagged products long enough. The above equation is a linear equation, and it is the basis for lag-profile inversion (Virtanen et al., 2008, Nikoukar et al., 2008).

Lag-profile inversion

Maximum likelihood estimate of autocorrelation function estimates:

$$\hat{\boldsymbol{\sigma^\tau}} = (\overline{\mathbf{W}^\tau}^T \mathbf{\Sigma^{-1}} \mathbf{W}^\tau)^{-1} \overline{\mathbf{W}^\tau}^T \mathbf{\Sigma^{-1}} \mathbf{m}^{\boldsymbol{\tau}}. \tag{26}$$

We can also look at the "error bars", or the covariance matrix of our estimate:

$$\mathbf{\Sigma_p} = (\overline{\mathbf{W}^\tau}^T \mathbf{\Sigma^{-1}} \mathbf{W}^\tau)^{-1}, \tag{27}$$

where $\mathbf{\Sigma} = \mathrm{E}\boldsymbol{\xi'}\overline{\boldsymbol{\xi'}}^T$ is the covariance matrix of the errors.

Lag-profile inversion

Useful features:

- The framework of linear inverse problems can be applied to solve the problem (a lot of classical results available to analyze the problem)
- If coding is good enough, the problem is not ill-posed, and we get unbiased estimates of the range dependent autocorrelation functions.
- Missing measurements are not a problem (eg., outlier removal can be done in receiver sample level and it doesn't bias our estimates).
- We can use different range resolutions for different altitudes

Lag-profile inversion

Problematic features:

- Poor coding results in an ill-posed problem and regularization is needed, which biases our results.
- With high signal to noise ratio, the assumption of circular symmetric Gaussian errors is wrong and the use of the wrong covariance degrades the results. This could be fixed by using generalized complex Gaussian errors with $\mathrm{E}\boldsymbol{\xi'}\boldsymbol{\xi'}^T \neq \mathbf{0}$, but this is dependent of our unknown radar target.
- Ok for single beam applications, but matrix equations are slow for high resolution multi-beam analysis.

Fast Lag-profile inversion

If we now rewrite some of the terms in more compact form ($m_t\overline{m_{t+\tau}} = m_t^\tau$, $\epsilon_{t-r}\overline{\epsilon_{t-r+\tau}} = \epsilon_{t-r}^\tau$, and $\mathrm{E}\zeta_{r,t}\zeta_{r,t+\tau} = \sigma_r^\tau$), we obtain:

$$m_t^\tau = \sum_r \epsilon_{t-r}^\tau \sigma_r^\tau + \xi_t', \tag{28}$$

we notice that the equation is actually a convolution equation. Another way to view the problem is that each lag of the autocorrelation function is convolved with the range ambiguity function.

Fast Lag-profile inversion

Obtaining the maximum likelihood estimate for the equation

$$\mathbf{m}_{\mathbf{t}}^{\boldsymbol{\tau}} = \mathbf{W}^{\tau}\boldsymbol{\sigma}^{\tau} + \boldsymbol{\xi}', \tag{29}$$

would require calculating:

$$\hat{\mathbf{x}} = (\overline{\mathbf{W}^{\tau}}^{T}\boldsymbol{\Sigma}^{-\mathbf{1}}\mathbf{W}^{\tau})^{-1}\overline{\mathbf{W}^{\tau}}^{T}\boldsymbol{\Sigma}^{-\mathbf{1}}\mathbf{m}. \tag{30}$$

The Fast Fourier Transform (FFT) is your friend.

$$Z_m = \sum_{n=0}^{N-1} z_n e^{-i2\pi mn/N}, \tag{31}$$

- Spectral estimation
- Convolutions
- Deconvolutions
- Diagonalization of circulant form linear equations
- Grid search of the likelihood function of a moving point target model

So what can you do with a high power large aperture radar?

- Ionospheric plasma parameters
- Meteor radar (accurate orbital elements of mg sized meteoroids)
- Planetary radar (asteroids, planets, moons)
- Space debris tracking (down to mm diameters)
- Radar imaging (Field aligned irregularities)

Why work with high power large aperture radar?

- It is challenging. Once you know how to deal with incoherent scatter radars, all other types of radar measurements are easy.
- It is fun.