Incoherent Scatter Theory: A Slightly Deeper Look

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ISR School 2016
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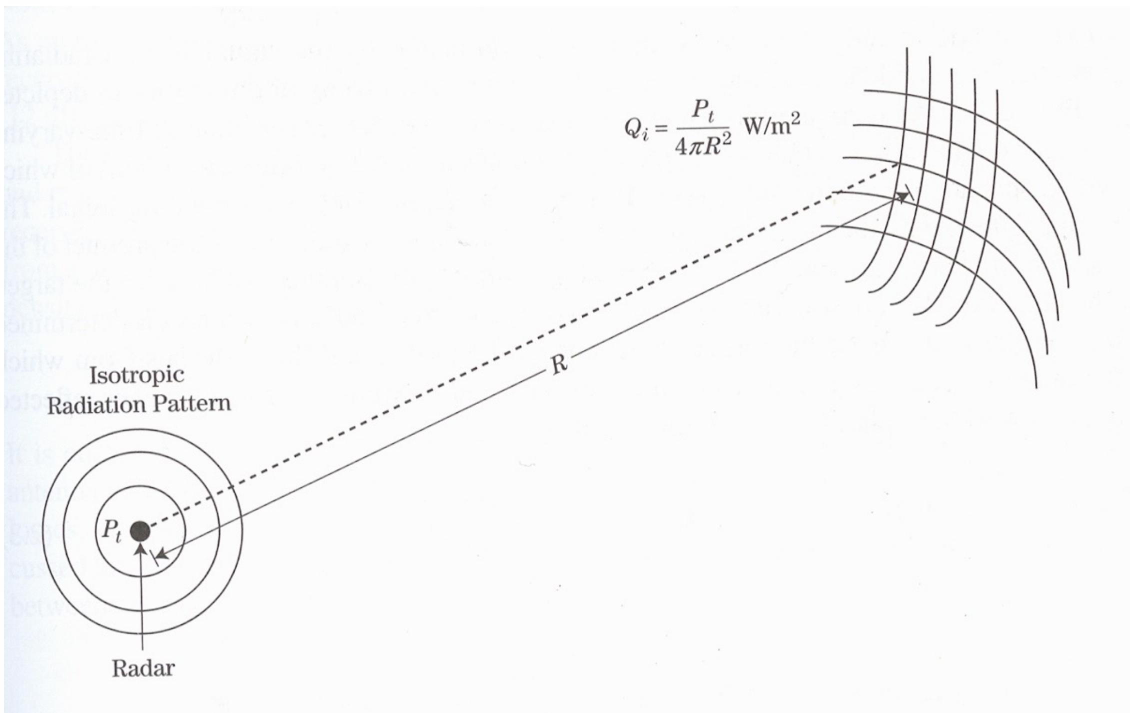
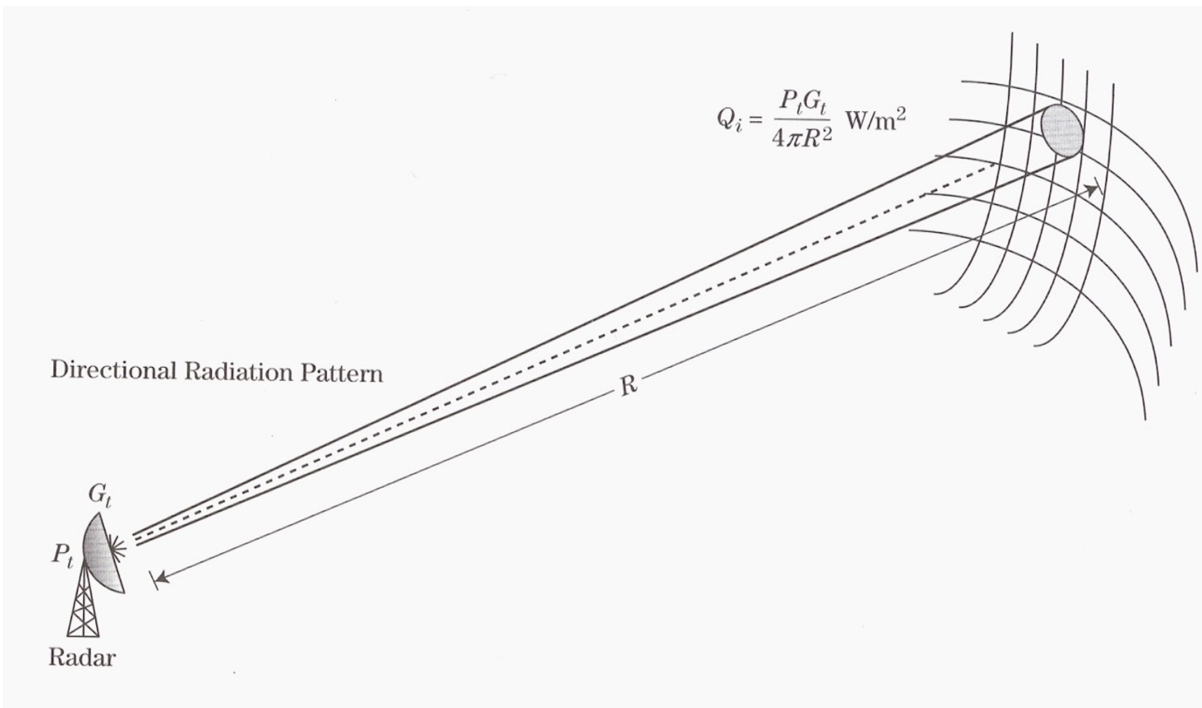
The Radar Equation: Monostatic Version

Power density at range R (isotropic): $$\frac{P_t}{4\pi R^2}$$



The Radar Equation: Monostatic Version

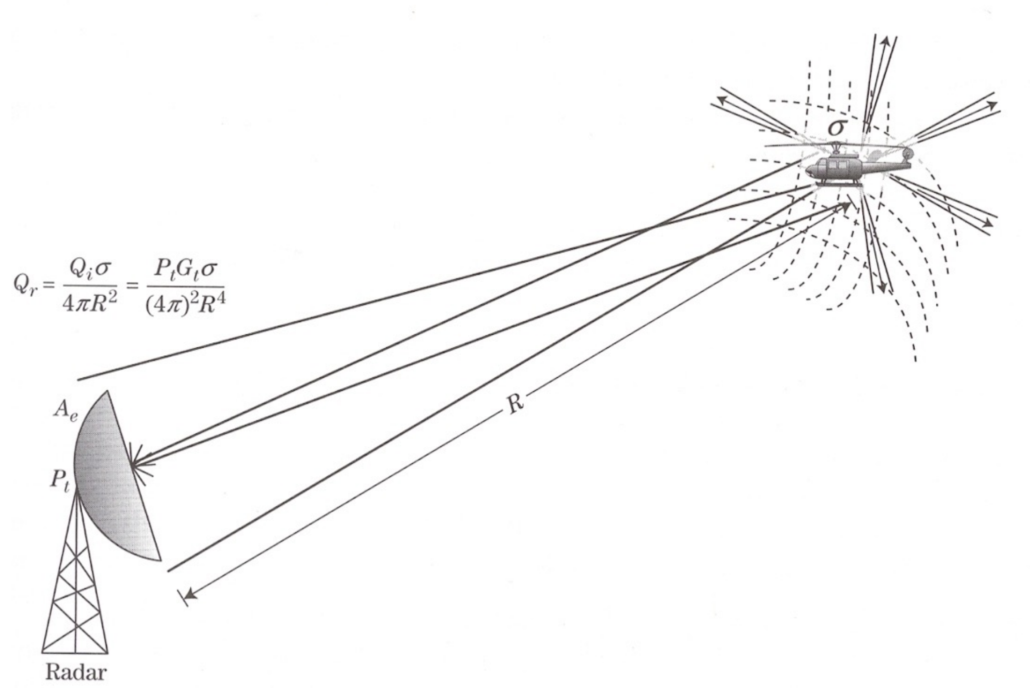
Power density at range R (directional): $$\frac{P_t\ G}{4\pi R^2}$$



The Radar Equation: Monostatic Version

Radar cross section (m^2)

Reradiated power density at Rx: $$\frac{P_t\ G}{4\pi R^2}\cdot\frac{\sigma}{4\pi R^2}$$



The Radar Equation: Monostatic Version

Total received power: $$P_r = \frac{P_t\ G}{4\pi R^2} \cdot \frac{\sigma}{4\pi R^2} \cdot A_e = \frac{P_t\ G\ A_e\ \sigma}{(4\pi)^2 R^4}$$

Use gain/area relation -

The Radar Equation:

$$P_r = P_t \frac{A_e^2}{4\pi\lambda^2 R^4}\sigma$$

Hard vs Soft Radar Targets

Generalize radar equation for one or more scatterers, distributed over a volume:

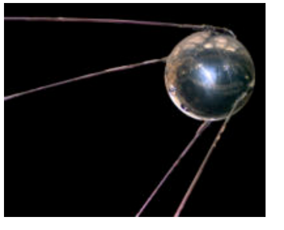
$$P_r = \int P_t \frac{A_e^2}{4\pi\lambda^2 R^4} \eta(\vec{x}) \, dV_s$$

First case: single scatterer ("hard target") at single point in space:

$$\int \eta(\vec{x}) \, dV_s = \sigma_{target} \equiv \sigma$$

Hard target radar equation:

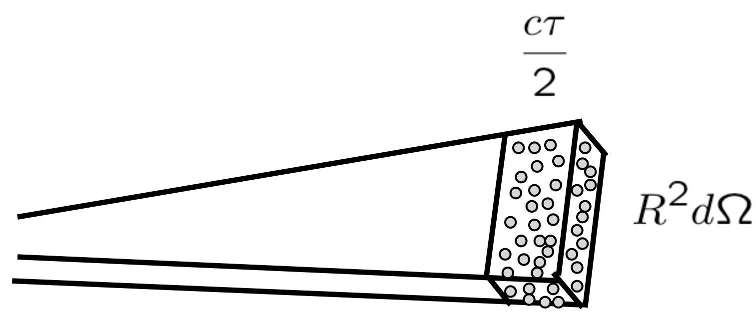
$$P_r = P_t \frac{A_e^2}{4\pi\lambda^2 R^4} \sigma$$



Sputnik 1 (1957-10-04)

Distributed Targets

$$\int \eta(\vec{x})\ dV_s = \int_0^{2\pi} \int_0^{\pi} \eta(\vec{x})\ \frac{c\tau}{2}\ R^2\ \sin\theta\ d\theta\ d\phi$$



Assume volume is filled with identical, isotropic scatters

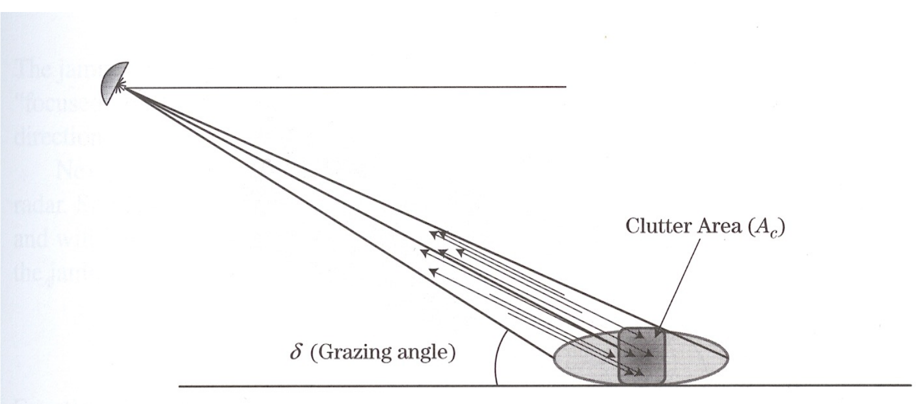
$$\int \eta(\vec{x})\ dV_s = \frac{c\tau}{2}\ R^2\ \eta_e$$

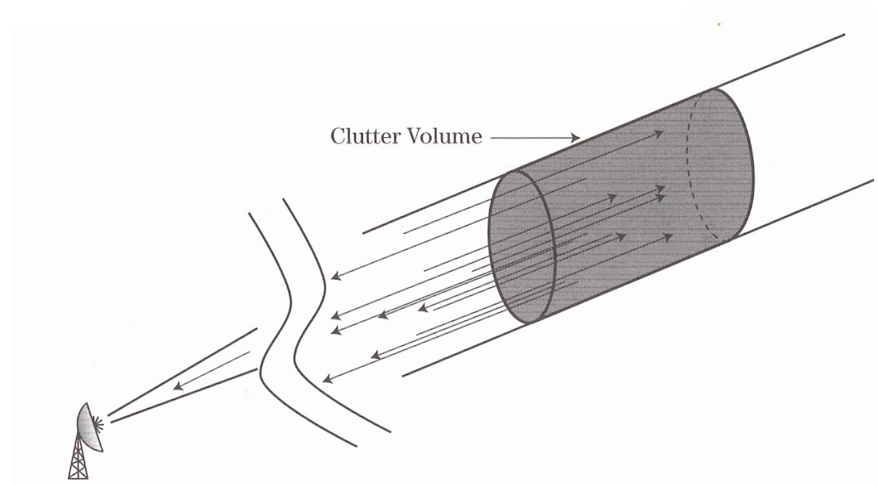
Distributed Scatterers

$$P_r = P_t \; \frac{A_e^2}{4\pi\lambda^2 R^4} \; \eta_e \; \frac{c\tau}{2} \; R^2$$

The "soft target" Radar Equation

$$P_r = P_t \; \frac{cA_e^2\tau}{8\pi\lambda^2 R^2} \; \eta_e$$

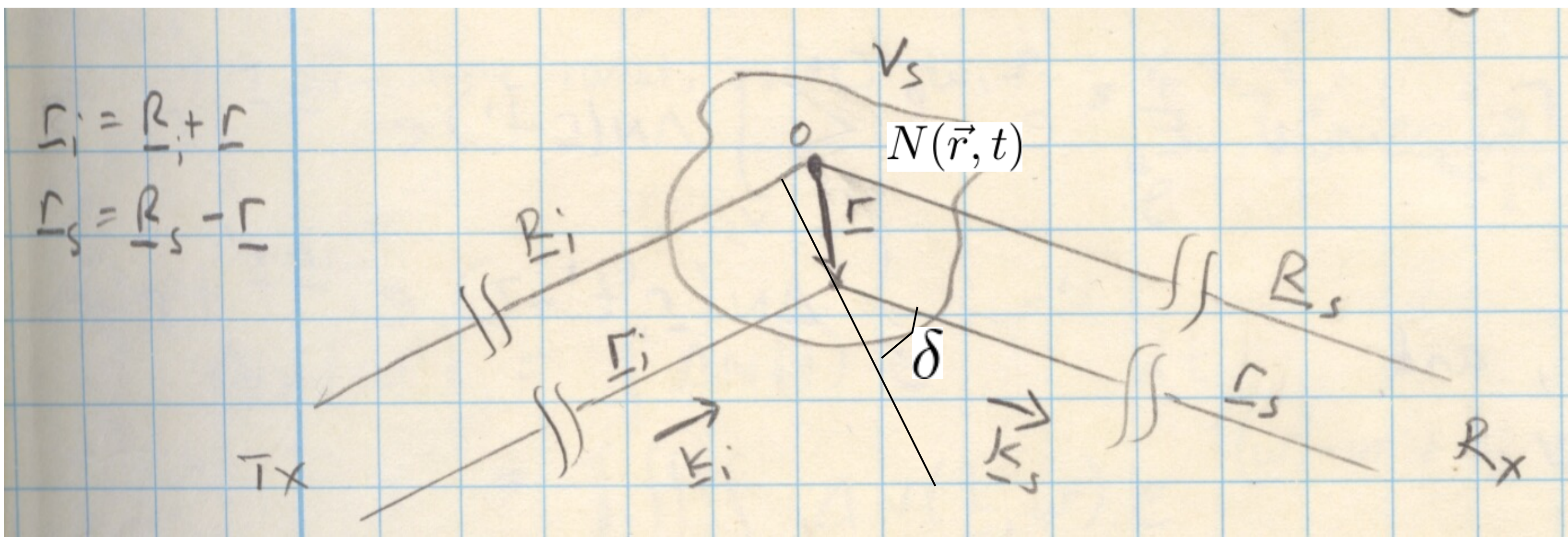




Single Electron Scattering Model

Incident EM wave accelerates each charged particle it encounters.
These then re-radiate an EM wave (as Hertzian dipoles).

For a single electron located at r = 0, we need the scattered field at a distance r_s.



Single Electron Scattering Model

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scattered field $$\left|\vec{E}_s(\vec{r}_s, t)\right| = \frac{e^2 \mu_0 \sin\delta}{4\pi r_s m_e}\left|\vec{E}_i(0, t^{'})\right|$$ Incident field

$$= \frac{r_e}{r_s}\sin\delta\left|\vec{E}_i(0, t')\right|$$

$$r_e = \frac{e^2 \mu_0}{4\pi m_e}$$ Classical electron radius

$$t' = t - \frac{r_s}{c}$$ Delayed time

$$\sin\delta$$ Scattering angle

Scattering Model

Assume a volume filled with electron scatterers whose density is represented in space and time by

$$N(\vec{r}, t)$$

Illuminating this volume with an incident field from a transmitter location means that each electron contributes to the resulting scattered field, using *Born approximation* (each scatter is weak and does not affect others).

With geometrical considerations, scattered field at receiver location is now:

$$E_s(t) = r_e \sin\delta \; E_0 e^{j\omega_0 t''} \int_{V_s} \frac{1}{r_s} N(\vec{r}, t') e^{-j(\vec{k_i} - \vec{k_s})\vec{r}} d^3\vec{r}$$

$$t'' = t' - \frac{r_i}{c}$$

Delayed time (TX to RX)

$$t' = t - \frac{r_s}{c}$$

Delayed time (volume to RX)

Scattering Model

Assume densities have random spatial and temporal fluctuations about a background:

$$N(\vec{r},t) \rightarrow N_0 + \Delta N(\vec{r},t)$$

Further, assume backscatter (i.e. monostatic radar):

$$\vec{k} = 2\vec{k}_i \qquad r_i \equiv r_s = R \qquad \sin\delta \rightarrow 1$$

Then, scattered field reduces to:

$$E_s(t) \rightarrow \frac{r_e}{R}\, E_0 e^{j\omega_0 t''} \underbrace{\int_{V_s} \Delta N(\vec{r},t') e^{-j\vec{k}\cdot\vec{r}} d^3\vec{r}}_{\equiv \Delta N(\vec{k},t')}$$

Maxwell's Equations

J. C. Maxwell
1831 - 1879

Governs propagation of electromagnetic waves ("action at a distance"), relation between electric and magnetic field and motions of charges
Foundation of classical electromagnetic theory

in free space:
H = B D = E

Gauss' Law (electric field around charges)

$$\nabla \cdot \mathbf{D} = \rho_f$$

Gauss' Law for magnetism (no magnetic monopoles)

$$\nabla \cdot \mathbf{B} = 0$$

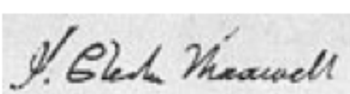
Faraday's Law (electric field around a changing magnetic field)

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Ampere's Law (magnetic field circulation around electric charges)

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

Maxwell's correction
(displacement current)



Scattering Model

Plasmas (ionosphere) are thermal gases and $\Delta N(\vec{r},t)$ is a Gaussian random variable, so the Central Limit Theorem applies:

statistical average

$$\langle E_s(t)\rangle = \langle \Delta N(\vec{r},t)\rangle = 0$$

It's much more useful to look at second order products – in other words, examine <u>temporal correlations</u> in the scattered field:

$$\langle E_s(t)\ E_s^*(t+\tau)\rangle \propto\ e^{-j\omega_0\tau}\left\langle \Delta N(\vec{k},t)\ \Delta N^*(\vec{k},t+\tau)\right\rangle$$

Useful things to measure can now be defined.

Scattering: Measurable Quantities

Defining $C_s = \frac{r_e^2 E_0^2 \sin^2 \delta}{R^2} V_s$, then

Total scattered power

$$\left\langle |E_s(t)|^2 \right\rangle = C_s \left\langle \left| \Delta N(\vec{k}) \right|^2 \right\rangle$$

and Autocorrelation function (ACF):

$$\langle E_s(t) E_s^*(t+\tau) \rangle = C_s e^{-j\omega_0 \tau} \left\langle \Delta N(\vec{k}, t) \Delta N^*(\vec{k}, t+\tau) \right\rangle$$

or Power Spectrum:

$$\left\langle |E_s(\omega_0 + \omega)|^2 \right\rangle \ \propto C_s \left\langle \left| \Delta N(\vec{k}, w) \right|^2 \right\rangle$$

Incoherent Scattering Model: Summary

Radar filters in k space:

$$\Delta N(\vec{r}, t) \rightarrow \Delta N(\vec{k}_r, t)$$

$$\Delta N(\vec{k}_r, t) \propto E_s(t)$$

Form ACF of $E_s(t)$ for each range, average, transform:

$$\langle E_s(t) E_s^*(t+\tau) \rangle \rightarrow \left\langle \left| \Delta N(\vec{k}, w) \right|^2 \right\rangle$$

Interpret latter in terms of the medium parameters.

Remote sensing a plasma: The experimental (radar) view

Suppose we transmit a wave towards a plasma and measure the scattered wave:

$$P_{rec} = (P_{inc}) A_{scat} (\frac{A_{rec}}{4\pi R^2})$$

$$A_{scat} = \sigma_{radar} V_s$$

(ionosphere is a beam filling target)

$$\sigma_{radar} = 4\pi\sigma_{total}$$

(Solid angle)

$$\left(\frac{P_{rec}}{P_{inc}}\right)\left(\frac{4\pi R^2}{A_{rec}}\right)\left(\frac{1}{V_s}\right) = 4\pi r_e^2 \sin^2\delta \left\langle |\Delta N(k)|^2 \right\rangle$$

Measurable experimentally

Physics info is here!

(1 for backscatter)

Radar cross-section of ionospheric plasma

Assume a beam filling plasma at F region altitudes (300 km) with very high electron density (1E12 electrons per m3 - BEST CASE):

Classical electron scattering cross-section $\sigma_e = 10^{-28} m^2/e^-$

Assume a pulse length of 10 km.
Assume a cross-beam width of 1 km (~ Arecibo).

Total cross section is then (10 km x 1 km x 1 km x 1E12 m^-3 x 1E-28 m^-2/e-):

$$\boxed{\sigma_{tot} \sim 10^{-6} m^2}$$

-60 dBsm! Are we going to be able to do this at all?

NB: Born approximation is very valid, since total amount of scattered power in the volume ~ 1E-12. So we can make full range profiles if we can detect the scatter.

Detectability of scatter from ionospheric plasma

For fraction of scattered power actually received, assume isotropic scatter and a BIG ~100 m diameter antenna:

$$f_{rec} = \frac{A_{rec}}{4\pi R^2} \sim \frac{10^4\ m^2}{4(300 \times 10^3\ m)^2}$$

About -80 dB (1E-8): not much. So:

$$\frac{P_{rec}}{P_{tx}} \sim 10^{-20}$$

So a radar with 1 MW transmitted signal receives 10 femtowatts of incoherently scattered power from free electrons in the ionosphere.

REALLY not very much.

Detectability of scatter from ionospheric plasma

What matters, though, is the signal to noise ratio:

$$P_{noise} = (k_B T_{eff})\,(BW)$$ (derived later)

Typical effective noise temperatures ~100 to 200 K at UHF frequencies (430 MHz, say).

Assume the bandwidth is set by thermal electron motions in a Boltzmann sense:

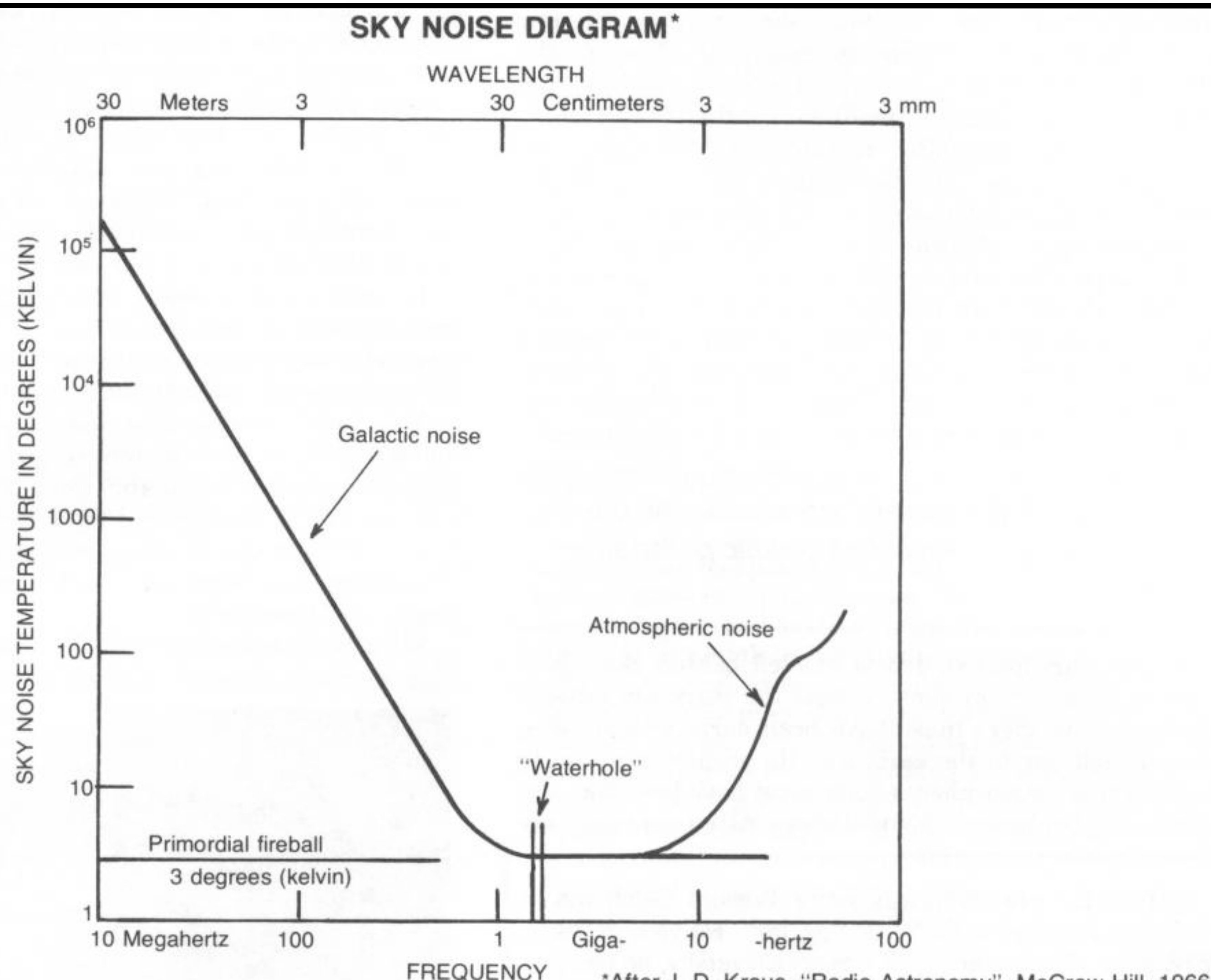
$$3k_B T_e \sim m_e v_{e,th}^2$$

$$v_{e,th} \sim \sqrt{\frac{3k_B T_e}{m_e}} \sim 2 \times 10^5 m/s$$

$$BW \sim (v_{e,th})\,(2)(2)(\frac{f_{tx}}{c}) \sim 10^6 Hz$$

(2s are for up/down, backscatter)

Sky Noise: The Universe Is Also Transmitting



*After J. D. Kraus, "Radio Astronomy", McGraw-Hill, 1966.

Nyquist-Johnson thermal noise (1928): Always present

JULY, 1928 *PHYSICAL REVIEW* *VOLUME 32*

THERMAL AGITATION OF ELECTRIC CHARGE IN CONDUCTORS*

BY H. NYQUIST

ABSTRACT

The electromotive force due to thermal agitation in conductors is calculated by means of principles in thermodynamics and statistical mechanics. The results obtained agree with results obtained experimentally.

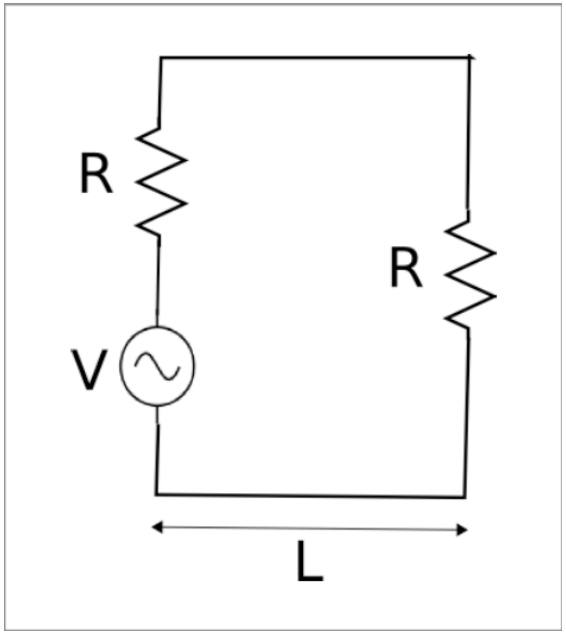
H. Nyquist 1889-1976
(born Nilsby, Sweden)

"Bert" Johnson 1887-1970
(born Gothenburg, Sweden)

Nyquist-Johnson: The Motivation

Nyquist-Johnson thermal noise (1928)

Circuit loop has a temperature T
(with thermal vibrations)



(Each resistor)

$$P_{absorbed} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \ \frac{h|\omega|}{e^{\frac{h|\omega|}{k_B T}} - 1}$$

$$I = \frac{V}{2R} \qquad P_{emitted} = \frac{\langle V^2 \rangle}{4R}$$

$S(\omega)$

Voltage spectral density in the circuit loop (= per frequency):

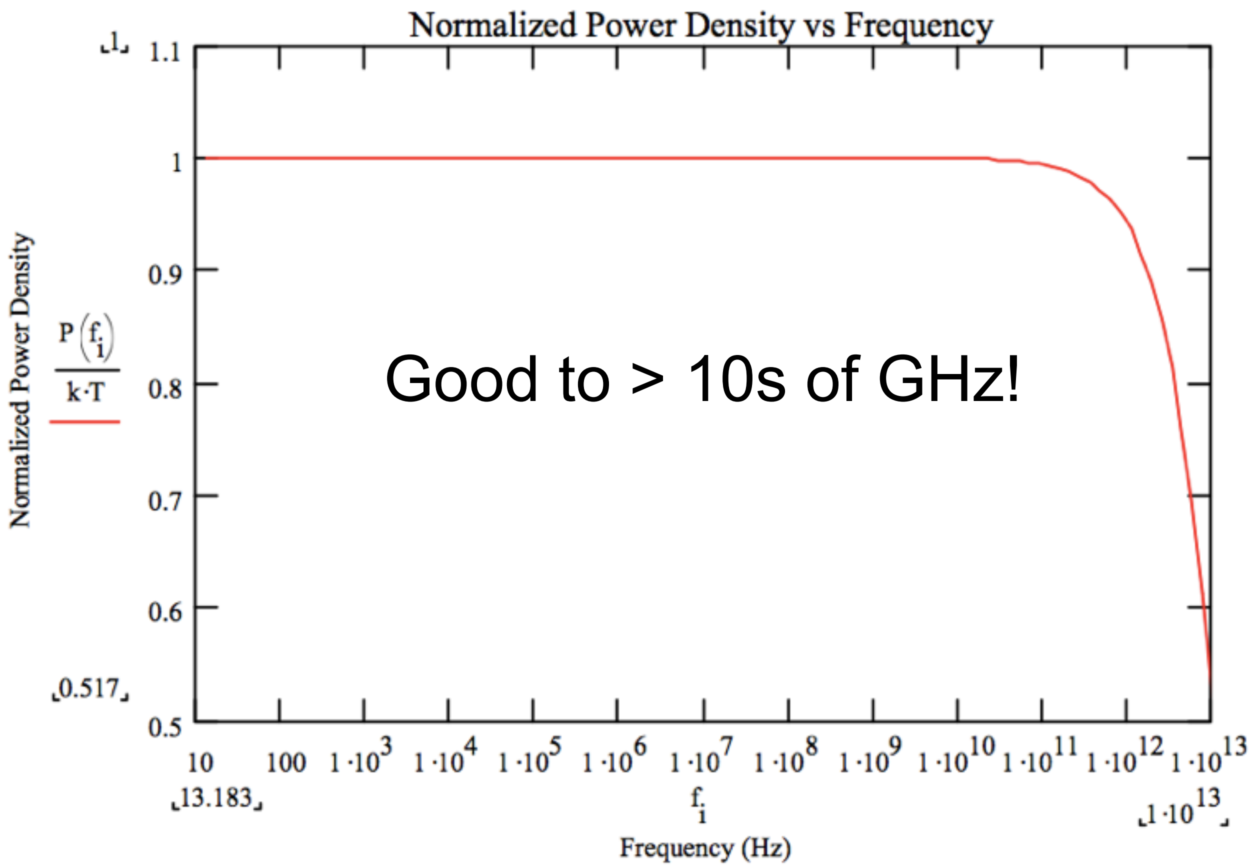
$$S(\omega) = 4R\frac{h|\omega|}{e^{\frac{h|\omega|}{k_B T}} - 1}$$

$$h\omega \ll k_B T : S(\omega) = 4Rk_B T$$

Over a range of frequencies:

$$\boxed{P(\Delta f) \propto k_B T \ \Delta f}$$

Nyquist-Johnson thermal noise (1928)



(Clay Turner, Wireless Solutions, 2007)

Detectability of scatter from ionospheric plasma

Finally,

$$P_{noise} \sim 2 \times 10^{-15} W$$

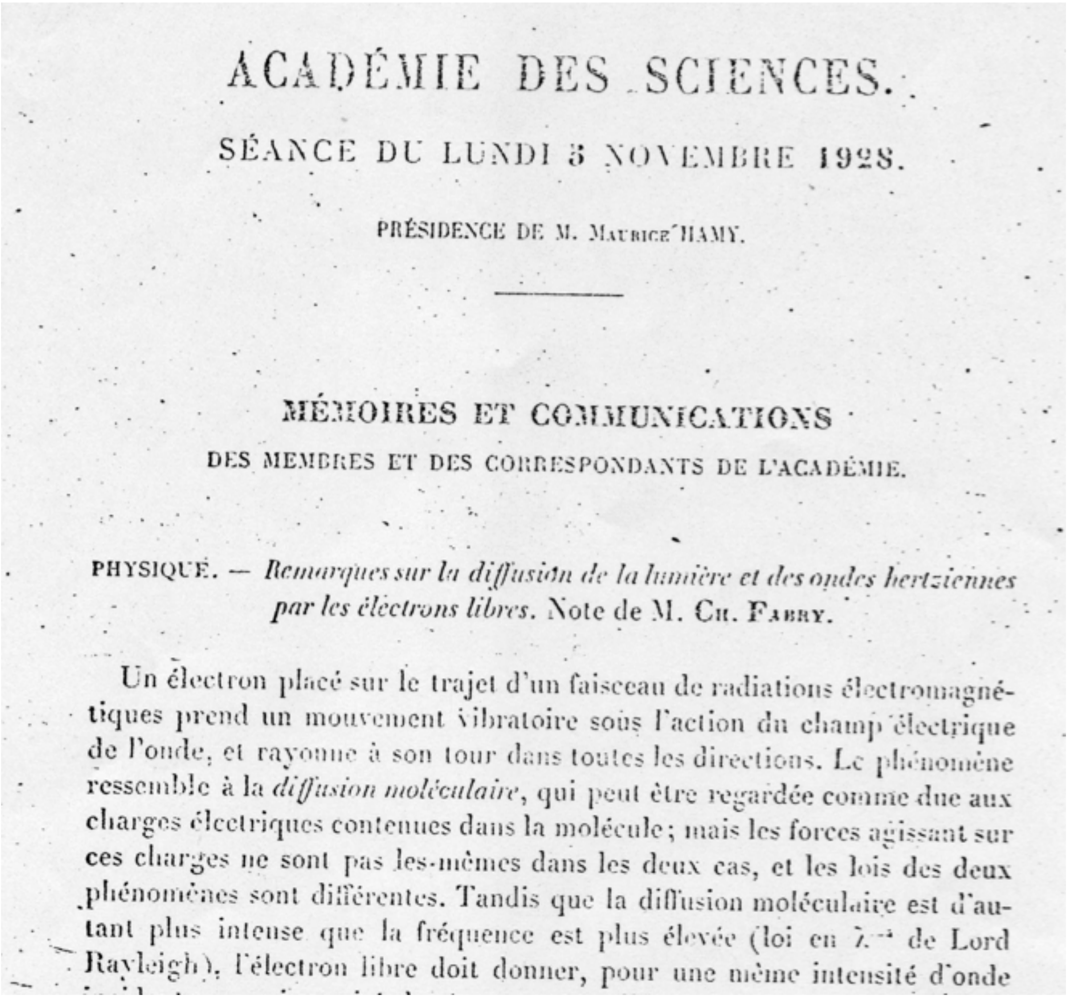
$$S/N \sim 5$$

Workable!

But you need a megawatt class transmitter and a huge antenna.

1950s: technology makes this possible (radio astronomy + construction = large antennas, military needs = high power transmitters)

Incoherent Scatter Concepts Are Older Than You Think



ACADÉMIE DES SCIENCES.

SÉANCE DU LUNDI 5 NOVEMBRE 1928.

PRÉSIDENCE DE M. MAURICE HAMY.

MÉMOIRES ET COMMUNICATIONS

DES MEMBRES ET DES CORRESPONDANTS DE L'ACADÉMIE.

PHYSIQUE. — *Remarques sur la diffusion de la lumière et des ondes hertziennes par les électrons libres.* Note de M. CH. FABRY.

Un électron placé sur le trajet d'un faisceau de radiations électromagnétiques prend un mouvement vibratoire sous l'action du champ électrique de l'onde, et rayonne à son tour dans toutes les directions. Le phénomène ressemble à la *diffusion moléculaire*, qui peut être regardée comme due aux charges électriques contenues dans la molécule; mais les forces agissant sur ces charges ne sont pas les mêmes dans les deux cas, et les lois des deux phénomènes sont différentes. Tandis que la diffusion moléculaire est d'autant plus intense que la fréquence est plus élevée (loi en λ^{-4} de Lord Rayleigh), l'électron libre doit donner, pour une même intensité d'onde

Remarques sur la diffusion de la lumière et des ondes hertziennes par les electrons libres

C. Fabry
1928

Charles Fabry
1867-1945

$$\sigma = \frac{8\pi}{3}\frac{e^4}{m^2}$$

Electron scattering cross section (fundamental)

Incoherent Scatter Concepts Are Older Than You Think

Without worrying about noise:

- Rayleigh scattering $\propto \lambda^4$ [why is the sky blue?]
- Incoherent scatter independent of wavelength
  - [but it's weak]

Remarques sur la diffusion de la lumière et des ondes hertziennes par les electrons libres

C. Fabry
1928

Charles Fabry
1867-1945

Incoherent Scatter concept!

For luminous radiations whose wavelength is very small, there is no phase relation between the elementary waves sent out by the different electrons of even a small volume and it is the intensities which add up. Thus, if a certain volume contains a total number of electrons n, then the power that it scatters is that transmitted by an area S = nσ. With the degrees of ionization that can actually exist, the scattering of light by electrons is always very slight. That is why it plays no appreciable role in the production of light in the diurnal sky *.

First Incoherent Scatter Radar

- W. E. Gordon of Cornell is credited with the idea for ISR.
- *"Gordon (1958) has recently pointed out that scattering of radio waves from an ionized gas in thermal equilibrium may be detected by a powerful radar."* (Fejer, 1960)
- Gordon proposed the construction of the Arecibo Ionospheric Observatory for this very purpose (NOT for radio astronomy as the primary application)

~40 megawatt-acres

- 1000' Diameter Spherical Reflector
  - 62 dB Gain
- 430 MHz line feed 500' above dish
- Gregorian feed
- Steerable by moving feed.

Incoherent Scattering of Radio Waves by Free Electrons with Applications to Space Exploration by Radar*

W. E. GORDON†, MEMBER, IRE

Proceedings of the IRE, November 1958

Introduction

FREE electrons in an ionized medium scatter radio waves incoherently so weakly that the power scattered has previously not been seriously considered. The calculations that follow show that this incoherent scattering, while weak, is detectable with a powerful radar. The radar, with components each representing the best of the present state of the art, is capable of:

1) measuring electron density and electron temperature as a function of height and time at all levels in the earth's ionosphere and to heights of one or more earth's radii;
2) measuring auroral ionization;
3) detecting transient streams of charged particles coming from outer space; and
4) exploring the existence of a ring current.

* Original manuscript received by the IRE, June 11, 1958; revised manuscript received, August 25, 1958. The research reported in this paper was sponsored by Wright Air Dev. Ctr., Wright-Patterson Air Force Base, O., under Contract No. AF 33(616)-5547 with Cornell Univ.

† School of Elec. Eng., Cornell Univ., Ithaca, N. Y.

First Incoherent-Scatter Radar

- **K.L. Bowles [Cornell PhD 1955]**, Observations of vertical incidence scatter from the ionosphere at 41 Mc/sec. *Physical Review Letters* 1958:

"*The possibility that incoherent scattering from electrons in the ionosphere, vibrating independently, might be observed by radar techniques has apparently been considered by many workers although seldom seriously because of the enormous sensitivity required…*"

First Incoherent-Scatter Radar

…*Gordon (*W.E. Gordon from Cornell*) recalled this possibility to the writer* [spring 1958; D. T. Farley] *while remarking that he hoped soon to have a radar sensitive enough to observe electron scatter in addition to various astronomical objects…"*

Bowles executed the idea - hooked up a large transmitter to a dipole antenna array in Long Branch Ill., took a few measurements.

VOLUME 1, NUMBER 12 PHYSICAL REVIEW LETTERS DECEMBER 15, 1958

Table I. Parameters of radar equipment used.

Operating frequency	40.92 Mc/sec
Peak pulse power	(4 to 6) × 10^6 watts
Pulse duration	(50 to 150) × 10^{-6} sec
Average power	4 × 10^4 watts maximum
Receiver bandwidth	10, 15, or 30 kc/sec
Antenna cross section	116 × 140 meters (1024 half-wave elements in phase above ground)
Antenna polarization	north-south
Calculated antenna gain	~35 decibels/isotropic

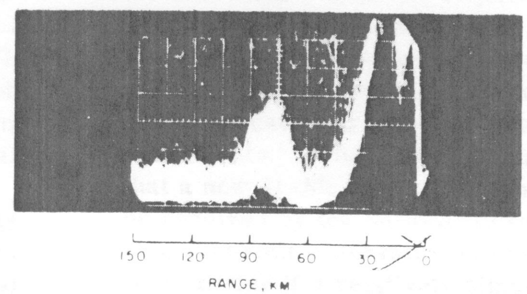


FIG. 2. Pulse width 50 μsec (~8 km); bandwidth 30 kc.

~6 week setup time

Oscilloscope + camera + ~4 sec exposure (10 dB integration)

Incoherent Scattering Detectability

Bowles' results found approximately the expected amount of power scattered from the electrons (scattering is proportional to charge to mass ratio - electrons scatter the energy).

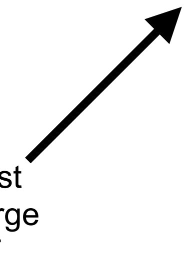
BUT: his detection with a 20 megawatt-acre system at 41 MHz (high cosmic noise background; should be marginal) implies a spectral width 100x narrower than expected – almost as if the much heavier (and slower) ions were controlling the scattering spectral width.

In fact, they do.

Dressed Test Particle Theory

Assumptions:
- Particles move in straight lines
- No B field (electrostatics)

$$\vec{x}(t) = \vec{x}_0 + \vec{v}_0 t$$



Charge density of one test charge T: $\rho_T(t) = q_T \ \delta(\vec{x} - \vec{x}_0 - \vec{v}_0 t)$

Everything starts with Vlasov equation:

$$\frac{D f_s}{Dt} = 0$$

Dressed Test Particle Theory

Vlasov, written out:

$$\frac{\partial f_s^{(1)}}{\partial t} + \vec{v} \cdot \frac{\partial f_s^{(1)}}{\partial t} + \frac{\vec{F}}{m_s} \cdot \frac{\partial f_{0,s}}{\partial \vec{v}} + \cdots = 0$$

$$\vec{F} = q_s \vec{E}$$

Electrostatic field:

$$\vec{E} = -\nabla \phi = \frac{\partial \phi}{\partial \vec{x}}$$

Substitute:

$$\frac{\partial f_s^{(1)}}{\partial t} + \vec{v} \cdot \frac{\partial f_s^{(1)}}{\partial t} - \frac{q_s}{m_s} \nabla \phi \cdot \frac{\partial f_{0,s}}{\partial \vec{v}} = 0$$

Dressed Test Particle Theory

We'll also need Poisson:

$$\nabla \cdot \vec{E} = -\nabla^2 \phi = \left(\sum_s \frac{q_s}{\epsilon_0} \int f_s^{(1)} d^3\vec{v} \right) + \frac{\rho_T}{\epsilon_0}$$

Perturbation in density of background plasma (partially neutralizing 'cloud')

test charge

Fourier transform in space:

$$\int \rho_T(\vec{x}, t)\, e^{-i\vec{k}\cdot\vec{x}}\, d^3\vec{x} = q_T\, e^{-i\vec{k}\cdot(\vec{x}_0 + \vec{v}_0 t)}$$

$$f_s^{(1)} = \frac{1}{(2\pi)^3} \int f_{s\vec{k}}^{(1)}\, e^{i\vec{k}\cdot\vec{x}} d^3\vec{k} \qquad f_{s\vec{k}}^{(1)} = \int f_s^{(1)}\, e^{-i\vec{k}\cdot\vec{x}} d^3\vec{x}$$

$$\phi = \frac{1}{(2\pi)^3} \int \phi_{\vec{k}}\, e^{i\vec{k}\cdot\vec{x}} d^3\vec{k} \qquad \phi_{\vec{k}} = \int \phi\, e^{-i\vec{k}\cdot\vec{x}} d^3\vec{x}$$

Dressed Test Particle Theory

Vlasov, Fourier transformed in space:

$$\frac{1}{(2\pi)^3}\int\left[\frac{\partial f^{(1)}_{s\vec{k}}}{\partial t} + i\vec{k}\cdot\vec{v}f^{(1)}_{s\vec{k}} - \frac{q_s}{m_s}\phi_{\vec{k}}\, i\vec{k}\cdot\frac{\partial f_{0,s}}{\partial\vec{v}}\right]e^{i\vec{k}\cdot\vec{x}}\,d^3\vec{k} = 0$$

Or

$$\frac{\partial f^{(1)}_{s\vec{k}}}{\partial t} + i\vec{k}\cdot\vec{v}f^{(1)}_{s\vec{k}} = \frac{iq_s}{m_s}\phi_{\vec{k}}\,\vec{k}\cdot\frac{\partial f_{0,s}}{\partial\vec{v}}$$

Dressed Test Particle Theory

Poisson equation, Fourier transformed in space:

$$k^2 \phi_{\vec{k}} - \sum_s \frac{q_s}{\epsilon_0} \int f^{(1)}_{s\vec{k}} d^3\vec{v} = \frac{q_T}{\epsilon_0}\, e^{-i\vec{k}\cdot(\vec{x}_0 + \vec{v}_0 t)}$$

Rewrite LHS in terms of a *relative* dielectric constant that takes into account the charge perturbation by the plasma ‘cloud’ as well as the test particle:

$$k^2 \phi_{\vec{k}} - \sum_s \frac{q_s}{\epsilon_0} \int f^{(1)}_{s\vec{k}} d^3\vec{v} \equiv k^2 \phi_{\vec{k}}\, \epsilon\left(\omega = \vec{k} \cdot \vec{v}_0, \vec{k}\right)$$

Dressed Test Particle Theory

So Poisson equation now takes the form

$$k^2 \phi_{\vec{k}}\, \epsilon \left(\omega = \vec{k} \cdot \vec{v}_0, \vec{k} \right) = \frac{q_T}{\epsilon_0}\, e^{-i\vec{k}\cdot(\vec{x}_0 + \vec{v}_0 t)}$$

where

$$\epsilon = 1 + \sum_s \epsilon_s \qquad \epsilon_s = \frac{-1}{k^2 \phi_{\vec{k}}} \frac{q_s}{\epsilon_0} \int f^{(1)}_{s\vec{k}} d^3\vec{v}$$

The physics of the medium is described by the dielectric constant (related to plasma conductivities)

Dressed Test Particle Theory

Let's push on - we want the perturbation dielectric constant.

Neglecting transients (which Landau damp out for long times), all the fluctuation terms must have a form in Fourier space similar to

$$\phi_{\vec{k}} \;\propto\; e^{-i\vec{k}\cdot(\vec{x}_0+\vec{v}_0 t)}$$

Vlasov:
$$\frac{\partial f^{(1)}_{s\vec{k}}}{\partial t} + i\vec{k}\cdot\vec{v}f^{(1)}_{s\vec{k}} = \frac{iq_s}{m_s}\phi_{\vec{k}}\,\vec{k}\cdot\frac{\partial f_{0,s}}{\partial\vec{v}}$$

Becomes
$$i\vec{k}\cdot(\vec{v}-\vec{v}_0)\,f^{(1)}_{s\vec{k}} = i\frac{q_s}{m_s}\phi_{\vec{k}}\vec{k}\cdot\frac{\partial f_{0,s}}{\partial\vec{v}}$$

Dressed Test Particle Theory

Now we can solve for the dielectric perturbation from the partially neutralizing 'cloud' around the test particle - recall:

$$\epsilon_s = \frac{-1}{k^2 \phi_{\vec{k}}} \frac{q_s}{\epsilon_0} \int f_{s\vec{k}}^{(1)} d^3\vec{v}$$

Vlasov eqn:

$$i\vec{k} \cdot (\vec{v} - \vec{v}_0)\, f_{s\vec{k}}^{(1)} = i \frac{q_s}{m_s} \phi_{\vec{k}} \vec{k} \cdot \frac{\partial f_{0,s}}{\partial \vec{v}}$$

'Cloud' around each test particle

$$\boxed{\epsilon_s = -\frac{q_s^2}{k^2 \epsilon_0 m_s} \int \frac{\vec{k} \cdot \frac{\partial f_{0,s}}{\partial \vec{v}}}{\vec{k} \cdot (\vec{v} - \vec{v}_0)}\, d^3\vec{v}}$$

Dressed Test Particle Theory

So now we can compute the potential and the dielectric constant (including perturbations from ‘clouds’ around test particles). These are collective effects and couple together all the particles.

Use this to find the electron density fluctuations in the medium (what we care about for scattering purposes). Now all sums are over individual particles.

Why only electrons?

$$\Delta N_e(\vec{k}, t) = \sum_{\text{ions, electrons}} [\text{electron clouds around test particles}]$$

$$+ \sum (\text{test electrons})$$

Dressed Test Particle Theory

$$\text{each } e^{-}\text{cloud} = \int f_{e\vec{k}}\, d^3\vec{v} = \left(\frac{\epsilon_0 k^2 \phi_{\vec{k}}}{e}\right) \epsilon_e \left(\omega = \vec{k}\cdot\vec{v}_0, \vec{k}\right)$$

(we used $\epsilon_s = \dfrac{-1}{k^2 \phi_{\vec{k}}} \dfrac{q_s}{\epsilon_0} \displaystyle\int f^{(1)}_{s\vec{k}} d^3\vec{v}$)

Poisson from some slides back:

$$k^2 \phi_{\vec{k}}\, \epsilon \left(\omega = \vec{k}\cdot\vec{v}_0, \vec{k}\right) = \frac{q_T}{\epsilon_0} e^{-i\vec{k}\cdot(\vec{x}_0 + \vec{v}_0 t)}$$

Here's the potential

Total dielectric constant from all electrons (clouds + test particles)

Dressed Test Particle Theory

So, add up all clouds plus test electrons:

$$\Delta N_e(\vec{k},t) = \sum_{\text{ions, electrons}} \frac{\epsilon_e q_j}{e\ \epsilon} e^{-i\vec{k}\cdot(\vec{x}_{0j}+\vec{v}_{0j}t)} + \sum_{\text{electrons}} e^{-i\vec{k}\cdot(\vec{x}_{0j}+\vec{v}_{0j}t)}$$

Total dielectric constant from all electrons (clouds + test particles)

Assume all particles are singly charged:

$$\Delta N_e(\vec{k},t) = \sum_{\text{ions}} \frac{\epsilon_e(\vec{k}\cdot\vec{v}_{0j},\vec{k})}{\epsilon(\vec{k}\cdot\vec{v}_{0j},\vec{k})} e^{-i\vec{k}\cdot(\vec{x}_{0j}+\vec{v}_{0j}t)} + \sum_{\text{electrons}} \left[1 - \frac{\epsilon_e(\vec{k}\cdot\vec{v}_{0j},\vec{k})}{\epsilon(\vec{k}\cdot\vec{v}_{0j},\vec{k})}\right] e^{-i\vec{k}\cdot(\vec{x}_{0j}+\vec{v}_{0j}t)}$$

Dressed Test Particle Theory

But these are random fluctuations. We want the ensemble average of the particle time correlations - this has information.

$$\left\langle \Delta N_e(\vec{k},t)\ \Delta N_e^*(\vec{k},t') \right\rangle$$

Key insight: we assume the positions of each particle are uncorrelated - but the Coulomb force does influence their TIME behavior.

$$\left\langle e^{-i\vec{k}\cdot\vec{x}_{0m}}\ e^{i\vec{k}\cdot\vec{x}_{0n}} \right\rangle = 0 \text{ for } m \neq n$$

So

$$\left\langle \Delta N_e(\vec{k},t)\ \Delta N_e^*(\vec{k},t') \right\rangle = \sum_{\text{ions}} \left| \frac{\epsilon_e}{\epsilon} \right|^2 e^{-i\vec{k}\cdot\vec{v}_{0j}(t-t')} + \sum_{\text{electrons}} \left| 1 - \frac{\epsilon_e}{\epsilon} \right|^2 e^{-i\vec{k}\cdot\vec{v}_{0j}(t-t')}$$

Dressed Test Particle Theory

We know each species (not particle) has its own distribution function - let's re-sort by species and write this expression replacing the particle sums with integrals:

$$\left\langle \Delta N_e(\vec{k},t)\; \Delta N_e^*(\vec{k},t') \right\rangle \;=\left[\sum_{\text{ion species}} \int \left|\frac{\epsilon_e}{\epsilon}\right|^2 f_{0s}(\vec{v}) e^{-i\vec{k}\cdot\vec{v}(t-t')} d^3\vec{v}\right] + \left[\int \left|1-\frac{\epsilon_e}{\epsilon}\right|^2 f_{0e}(\vec{v}) e^{-i\vec{k}\cdot\vec{v}(t-t')} d^3\vec{v}\right]$$

Notice only the time difference matters, so set

$$\tau = t - t'$$

and Fourier transform in time. **(We're almost there!)**

Dressed Test Particle Theory

$$\left\langle \left| \Delta N_e(\vec{k},\omega) \right|^2 \right\rangle = \left[\sum_{\text{ion species}} \int \int \left| \frac{\epsilon_e}{\epsilon} \right|^2 f_{0s}(\vec{v}) e^{-i\left(\vec{k}\cdot\vec{v}-\omega\right)\tau} d^3\vec{v}\, d\tau \right] + \left[\int \left| 1 - \frac{\epsilon_e}{\epsilon} \right|^2 f_{0e}(\vec{v}) e^{-i\left(\vec{k}\cdot\vec{v}-\omega\right)\tau} d^3\vec{v}\, d\tau \right]$$

becomes

$$\left\langle \left| \Delta N_e(\vec{k},\omega) \right|^2 \right\rangle = \left[\sum_{\text{ion species}} \int \left| \frac{\epsilon_e(\omega{=}\vec{k}\cdot\vec{v},\vec{k})}{\epsilon(\omega{=}\vec{k}\cdot\vec{v},\vec{k})} \right|^2 f_{0s}(\vec{v})\ \delta(\omega - \vec{k}\cdot\vec{v}) d^3\vec{v} \right] + \left[\int \left| 1 - \frac{\epsilon_e(\omega{=}\vec{k}\cdot\vec{v},\vec{k})}{\epsilon(\omega{=}\vec{k}\cdot\vec{v},\vec{k})} \right|^2 f_{0e}(\vec{v})\ \delta(\omega - \vec{k}\cdot\vec{v}) d^3\vec{v} \right]$$

Dressed Test Particle Theory

Assuming 1D motion along k direction, power spectrum:

$$\left\langle \left| \Delta N_e(\vec{k},\omega) \right|^2 \right\rangle = \left[\sum_{\text{ion species}} \left| \frac{\epsilon_e(\omega,\vec{k})}{\epsilon(\omega,\vec{k})} \right|^2 f_{0s}(\omega/k) \right] + \left[\left| 1 - \frac{\epsilon_e(\omega,\vec{k})}{\epsilon(\omega,\vec{k})} \right|^2 f_{0e}(\omega/k) \right]$$

Proceed (not done here) with assuming a Maxwellian distribution and deriving the dielectric constants.

In general, we find that

We need this

$$\epsilon(\omega,\vec{k}) = \text{function}\left(\omega^2/k^2\right)$$

Dispersion relation: the concept

We need this

$$\epsilon(\omega, \vec{k}) = \text{function}\left(\omega^2/k^2\right)$$

Dispersion relation = relationship between oscillation frequency and wave vector.

Each governing physical process in the wave medium - plasma in this case - has its own set of equations which define this relationship.

Simple case: uniform phase velocity

$$\omega(k) = c\ k$$

Most propagation speeds depend nonlinearly on the wavelength and/or frequency.

NB: for a nonlinear dispersion relation, the pulse will typically spread in either spatial frequency or temporal frequency as a function of time (dispersion).

Important thermal plasma dispersion relations

$$\epsilon(\omega, \vec{k}) = \text{function}\left(\omega^2/k^2\right)$$

Insert plasma dispersion relation here

We need the full dispersion relation expression.

This is not a plasma waves course so we won't derive it, but the two most important modes are:

1) *Ion-acoustic fluctuations* [sound waves in plasma]

$$\frac{\omega}{k} = \sqrt{\frac{k_B T_e + \gamma_i k_B T_i}{m_i}} = V_s$$

(Note no gamma term for electrons since they are isothermal, but ions are slow and suffer 1D compressions so their gamma term = 3)

NB: ordinary acoustic waves: adiabatic compression / decompression of fluid particles.

Ion-acoustic fluctuations: restoring force = electromagnetic

Important thermal plasma dispersion relations

$$\epsilon(\omega, \vec{k}) = \text{function}\left(\omega^2/k^2\right)$$

Insert plasma dispersion relation here

We need the full dispersion relation expression.

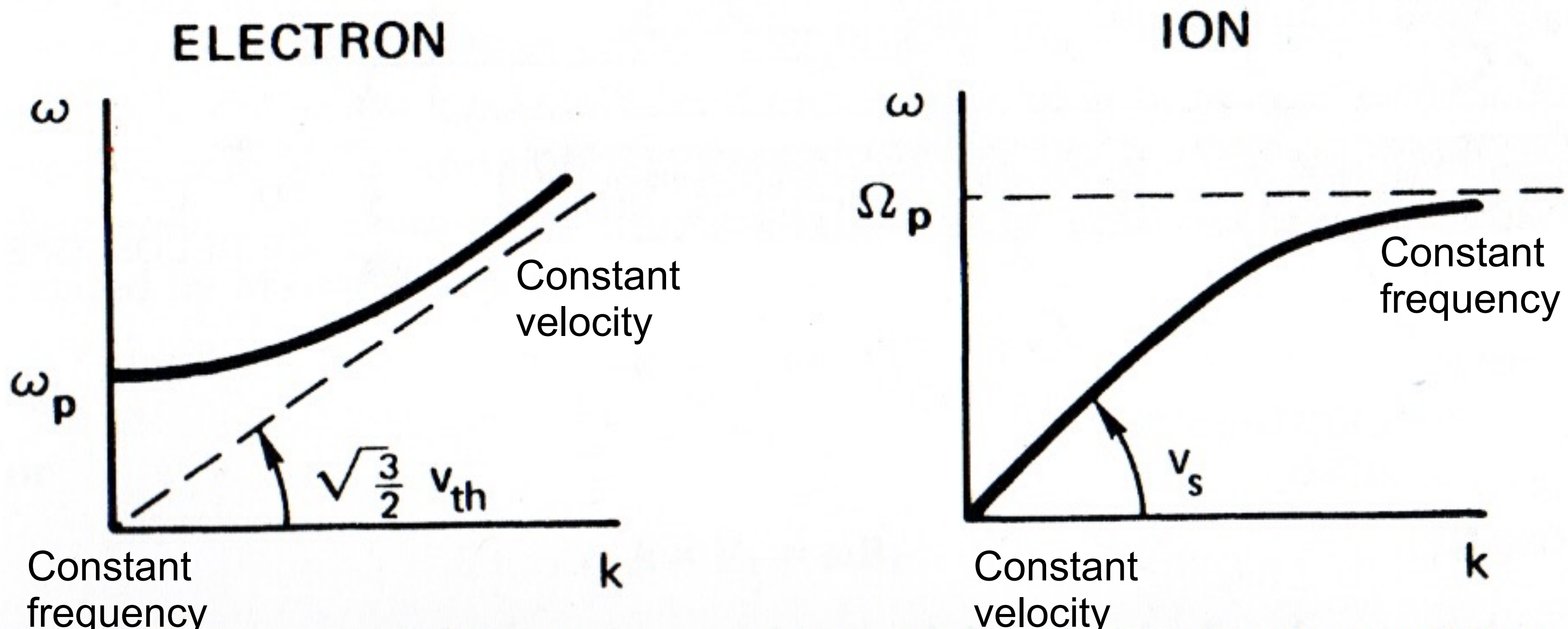
This is not a plasma waves course so we won't derive it, but the two most important modes are:

2) *Langmuir oscillations* (Plasma oscillations):

$$\omega^2 = \omega_p^2 + \frac{3}{2}k^2 v_{th}^2 \quad v_{th}^2 = 2k_B T_e/m_e$$

Akin to Brunt-Våisålå oscillations in fluid (parcel in presence of density gradient) - here, electrostatic field is restoring force, and electron pressure gradient transmits information

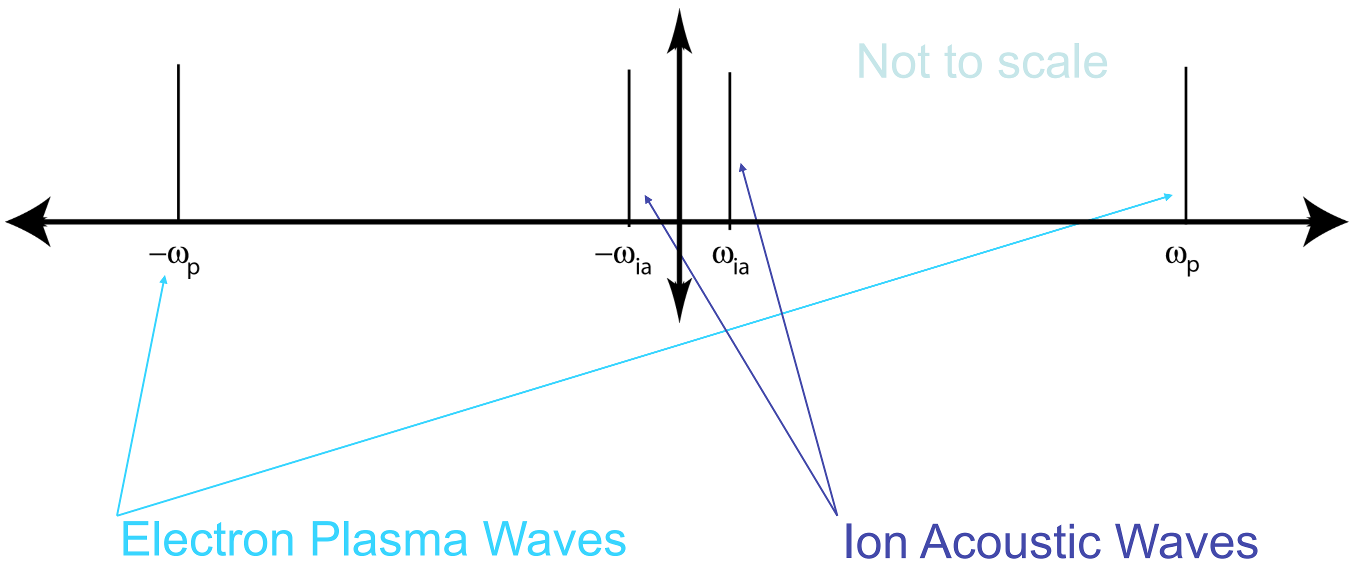
Electron and ion waves: Dispersion relations



Comparison of the dispersion curves for electron plasma waves and ion acoustic waves.

(Chen, Intro to Plasma Physics)

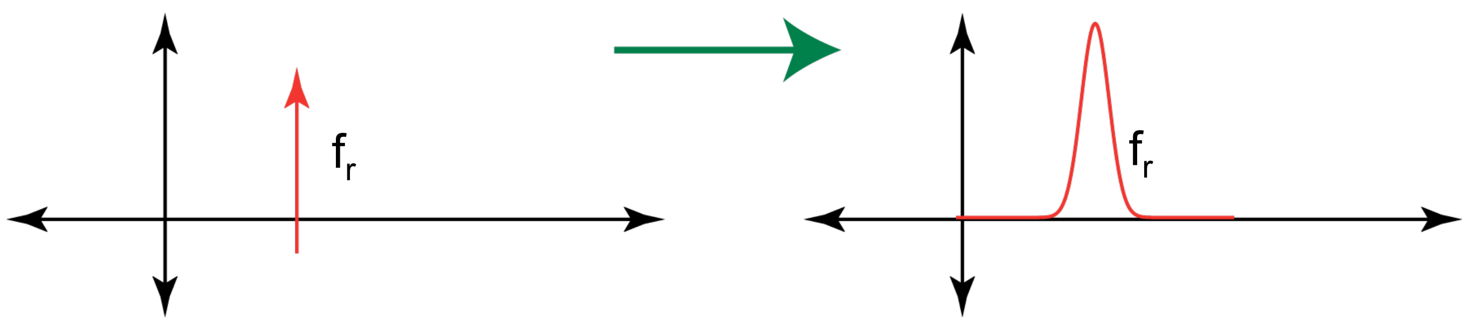
Wave Spectrum



Plasma parameters fluctuate with the waves (density, velocity, etc)

Damped resonance

- Waves in a plasma are resonances.
- Damped resonances are not sharp
  - Example – Q of a resonant circuit.
- IS: Thermal ions have motions close to ion-acoustic speed (Landau damping – “surfing”; locked to I-A waves)



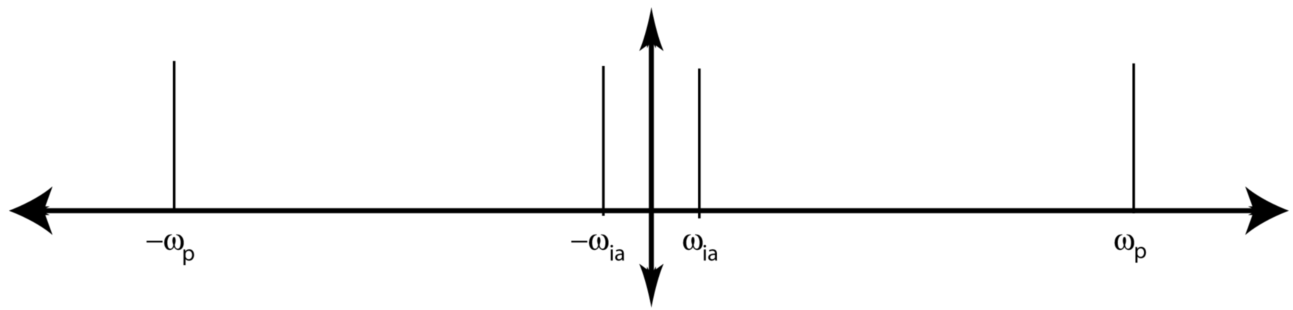
Resonance

Damped Resonance

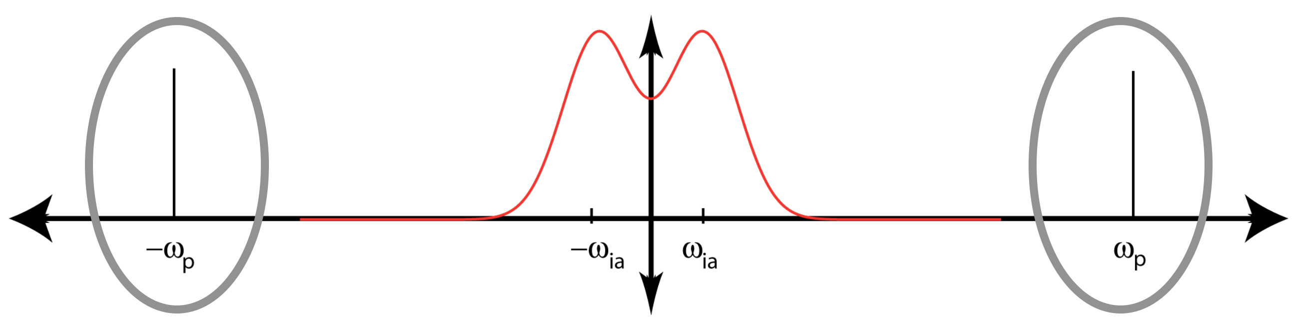
Wave Spectrum (ISR Spectrum)

UHF center freq:
10-20 kHz wide

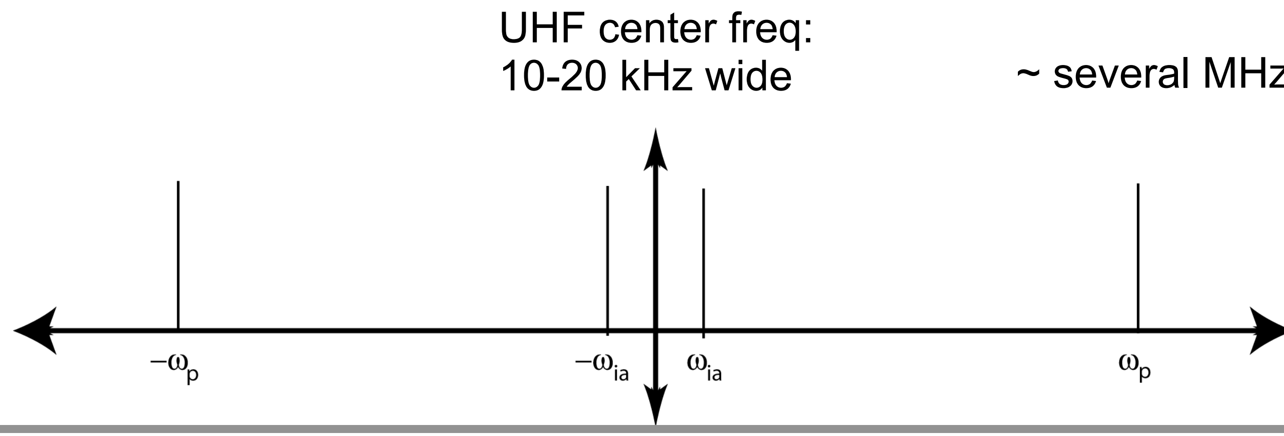
~ several MHz



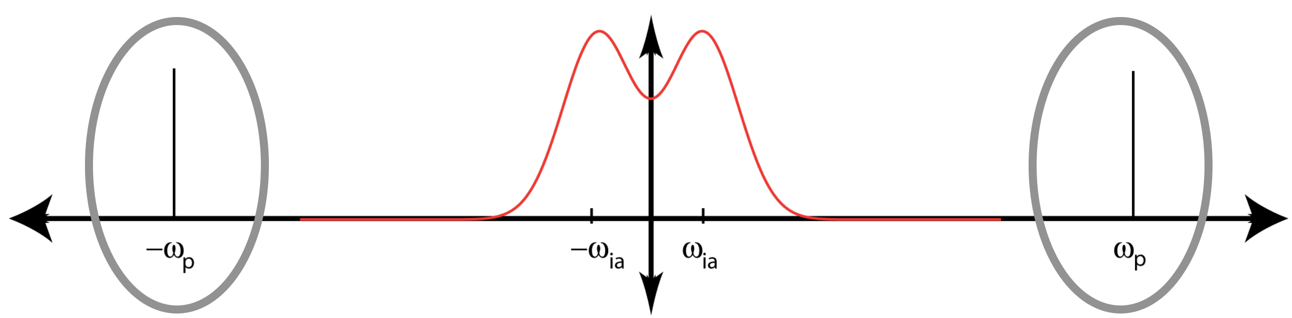
Why aren't the Langmuir (plasma) waves damped?



Wave Spectrum (ISR Spectrum)

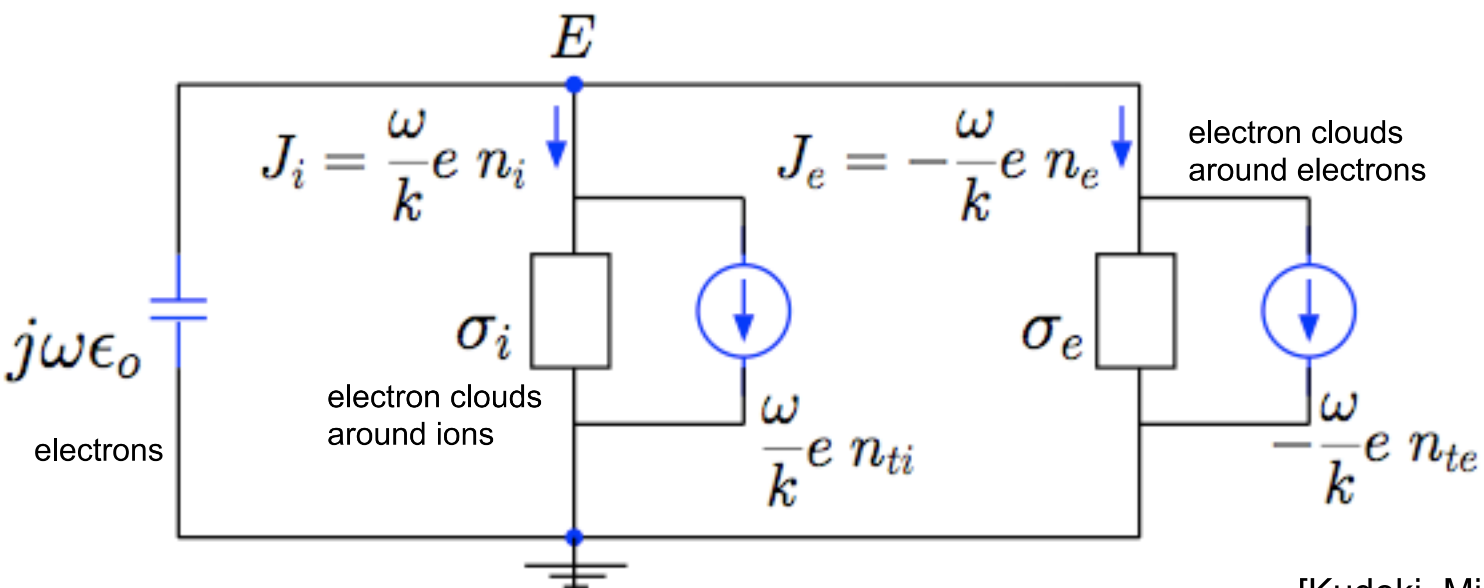


Why aren't the Langmuir (plasma) waves damped?
Electron thermal velocity ~ 125 km/s but plasma wave frequency ~ several MHz –
Not much interaction and not much damping.



Incoherent Scattering Model

$$\sigma_0(\omega_o+\omega)d\omega = N_0 r_e^2 \,\mathrm{Re}\left\{ \frac{y_e(y_i + jk^2\lambda_{de}^2)}{y_e + y_i + jk^2\lambda_{de}^2}\,\frac{d\omega}{\pi\omega} \right\}$$



[Kudeki, Milla]

Incoherent Scattering Model

$$\sigma_0(\omega_o + \omega)d\omega = N_0 r_e^2 \,\mathrm{Re}\left\{ \frac{y_e(y_i + jk^2\lambda_{de}^2)}{y_e + y_i + jk^2\lambda_{de}^2} \frac{d\omega}{\pi\omega} \right\}$$

- Short wavelength limit ($k^2\lambda_{de}^2 >> 1$): pure e^- scatter
- Long wavelength limit: RHS $\rightarrow y_e y_i/(y_e + y_i)$: damped ion-acoustic resonances
- Near plasma frequency: $y_e + y_i + jk^2\lambda_{de}^2 \rightarrow 0$: plasma lines

Incoherent Scattering Model

Spectral response can be evaluated using these frameworks for:

- Thermal inequality $T_e \neq T_i$: decreases Landau damping
- Ion-neutral collisions ν_{in}: narrows spectrum
- Background magnetic field B_0: makes electrons heavier

$$m_e \longrightarrow m_e^* = \frac{m_e}{\cos^2 \alpha}$$

Also, ion gyro-resonance (mass-dependent).

Incoherent Scattering Model

- Ion mixtures: $\frac{T_e}{T_i} y_i \rightarrow \sum_j \frac{T_e}{T_j}\frac{N_j}{N_0} y_j(m_j, T_j)$
- Unequal ion temperatures
- Particle drifts: $\omega \rightarrow \omega - \vec{k}\cdot\vec{v}_{de}$
- Plasma line measurements $([e^-], T_e, v_\parallel)$
- Photoelectron heating, non-Maxwellian plasmas
- Faraday rotation effects (equator, low TX freq)

Things can get hairy. For example, magnetic field evaluation requires Gordeyev integral:

$$\int e^{j(\theta - j\phi)t - \frac{\sin^2\alpha}{\phi^2}\sin^2(\frac{\phi t}{2}) - \frac{t^2}{4}\cos^2\alpha}\, dt$$

Incoherent Scattering Model

(See IS Spectrum Java applet on “ISR Demonstration” page)