

# ISR Data Analysis and Fitting

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## 1 ISR Data Processing

- Power Estimation
- Underspread ACF Estimation
- Overspread ACF Estimation

## 2 ISR Fitting

- Parameter Estimation
- Limitations on ISR Fitting

## 3 Derived ISR Data Products

- Vector Winds and Electric Fields
- Other Derived Parameters

# ISR Data Processing

- ① Collect baseband voltage samples
- ② Form Power and ACF estimates (lag-products)
  - Requires averaging over many pulses
- ③ Fit ACFs for plasma parameters at each altitude using ISR theory
  - $N_e, T_e, T_i, V_{LOS}$
  - Composition? Collision frequencies?
- ④ Process plasma parameters into higher level derived data products
  - Vector electric fields
  - Conductivities
  - Joule heating
  - Particle precipitation characteristics

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# Power Estimation

Given  $K$  samples  $v_i = s_i + n_i$ , and an independently known noise power,  $N$

$$\hat{S} = \frac{1}{K} \sum_{i=0}^{K-1} v_i v_i^* - N$$

$$E \{ \hat{S} \} = S \quad \text{unbiased estimator}$$

$$\text{Var} \{ \hat{S} \} = \frac{1}{K} (S + N)^2$$

$$\frac{\delta \hat{S}}{S} = \frac{1}{\sqrt{K}} \left( 1 + \frac{1}{S/N} \right)$$

For example,  $\frac{\delta \hat{S}}{S} = 0.5$  with a  $S/N = 0.1$  requires  $K = 484$ .

This assumes the samples are taken far apart and are uncorrelated.

# Electron Density Determination

- ISR Power received (Watts)

$$P_{\text{Rx}} = P_{\text{Tx}} \frac{\tau_p}{2R^2} K_{\text{sys}} N_e \frac{2}{(1 + k^2 \lambda_{De}^2) (1 + k^2 \lambda_{De}^2 + T_e/T_i)}$$

$$\lambda_{De} = \sqrt{\frac{\epsilon_0 k_B T_e}{e^2 N_e}} \quad k = \frac{4\pi}{\lambda_{\text{Tx}}} \quad R = \text{Range} \quad \tau_p = \text{Pulse Length (s)}$$

- $P_{\text{Rx}}$  in Watts determined by comparing relative power received to direct signal injection (cal pulses)
- “System Constant”  $K_{\text{sys}}$  involves antenna gain, effective area, etc. For PFISR  $K_{\text{sys}} \sim 10^{-19} \text{ m}^5 \text{ s}^{-1}$ .
- $K_{\text{sys}}$  determined by comparing to absolute  $N_e$  measurements
  - Ionosonde  $f_{0F2}$
  - ISR plasma line frequency
  - Faraday rotation (e.g. Jicamarca)

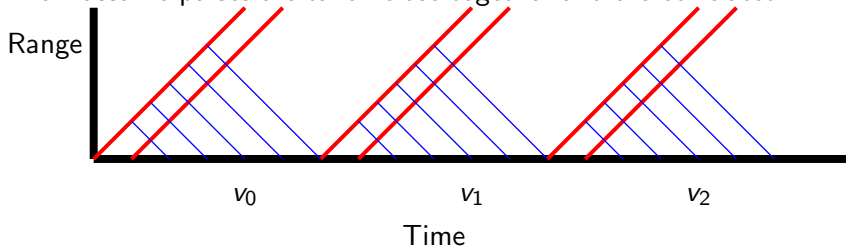
# Reporting Electron Density

$$\text{Temperature Correction: } \zeta = \frac{2}{(1 + k^2 \lambda_{De}^2) (1 + k^2 \lambda_{De}^2 + T_e/T_i)}$$

- Uncorrected  $N_e$ : Assume  $\zeta = 1$ .
  - $T_e/T_i = 1$
  - $k^2 \lambda_{De}^2 \ll 1$ .
- $N_e$  with model: Compute  $\zeta$  using an empirical model of  $T_e/T_i$  as a function of altitude.
- $N_e$  with fits: Compute  $\zeta$  with  $T_e$  and  $T_i$  estimated from fitted ACF.

# ACF Estimation (Pulse-to-Pulse)

Now assume pulses are taken close together and are correlated.



Unbiased Estimator:

$$\hat{R}_\ell = \frac{1}{K-\ell} \sum_{n=\ell}^{K-1} v_n v_{n-\ell}^*$$

$$E \left\{ \hat{R}_\ell \right\} = R_\ell$$

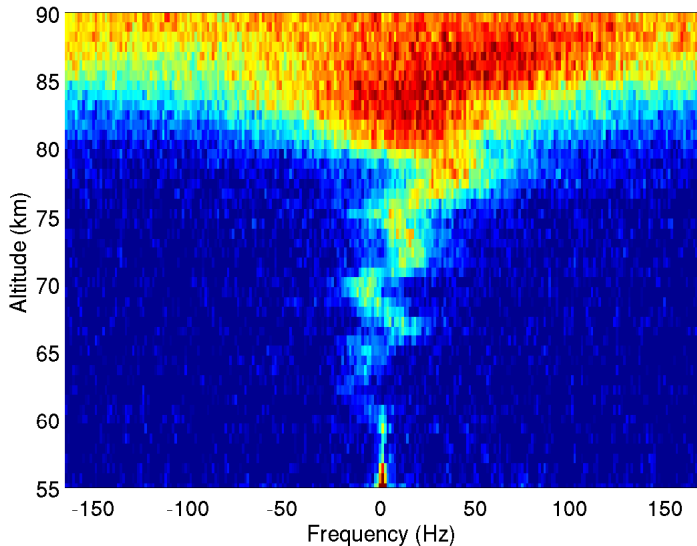
Biased Estimator:

$$\tilde{R}_\ell = \frac{1}{K} \sum_{n=\ell}^{K-1} v_n v_{n-\ell}^*$$

$$E \left\{ \tilde{R}_\ell \right\} = \frac{K-\ell}{K} R_\ell \quad \text{[triangular window]}$$

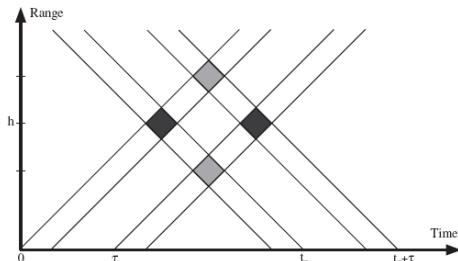
# Example D-region Spectra from PFISR

Typical D-region Spectra





# Double Pulse Experiment



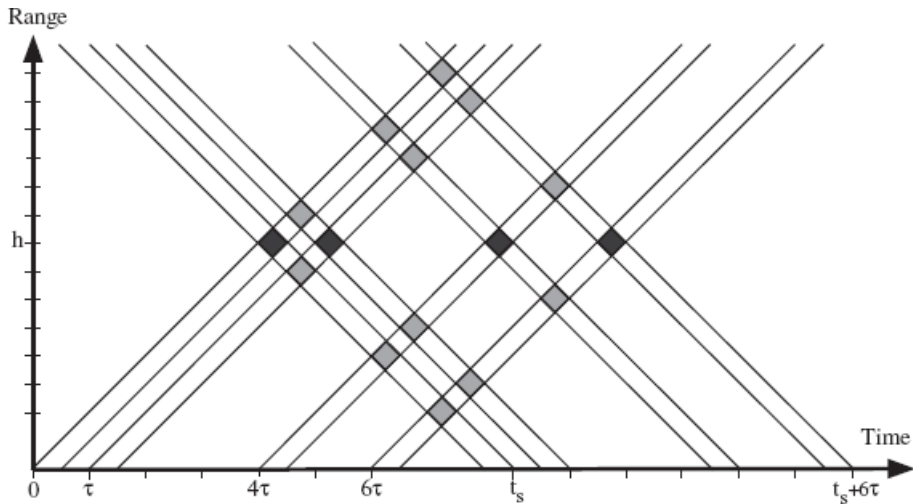
$$\begin{aligned}
 v(t_s + \tau)v^*(t_s) &= \left[ s(h; t + \tau) + s\left(h + \frac{c\tau}{2}; t + \frac{\tau}{2}\right) \right] \left[ s^*(h; t) + s^*\left(h - \frac{c\tau}{2}; t + \frac{\tau}{2}\right) \right] \\
 &= s(h; t + \tau)s^*(h; t) + s(h; t + \tau)s^*\left(h - \frac{c\tau}{2}; t + \frac{\tau}{2}\right) \\
 &\quad + s\left(h + \frac{c\tau}{2}; t + \frac{\tau}{2}\right)s^*(h; t) + s\left(h + \frac{c\tau}{2}; t + \frac{\tau}{2}\right)s^*\left(h - \frac{c\tau}{2}; t + \frac{\tau}{2}\right)
 \end{aligned}$$

ISR signals from disjoint altitudes are uncorrelated:

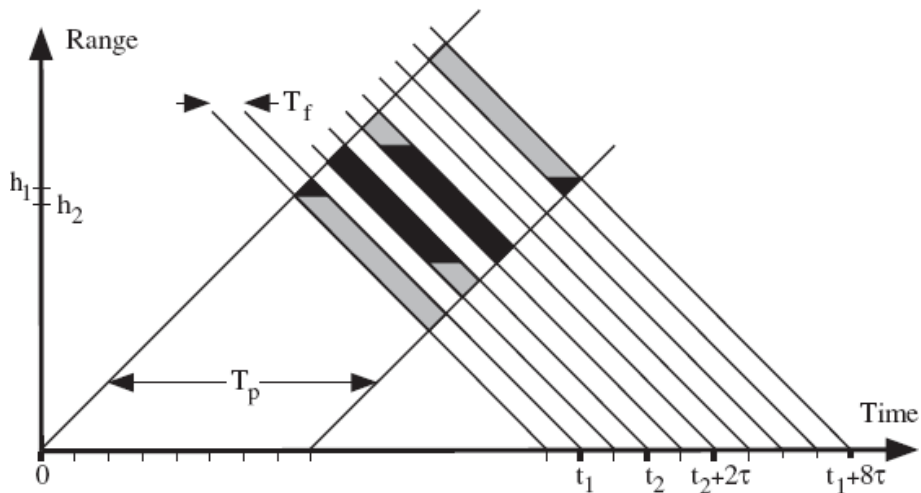
$$E \{ s(h_1; t + \tau) s^*(h_2; t) \} = \begin{cases} R(h_1; \tau) & h_1 = h_2 \\ 0 & h_1 \neq h_2 \end{cases}$$

$$E \{ v(t_s + \tau)v^*(t_s) \} = E \{ s(h; t + \tau) s^*(h; t) \} = R(h; \tau)$$

# Multipulse Experiments

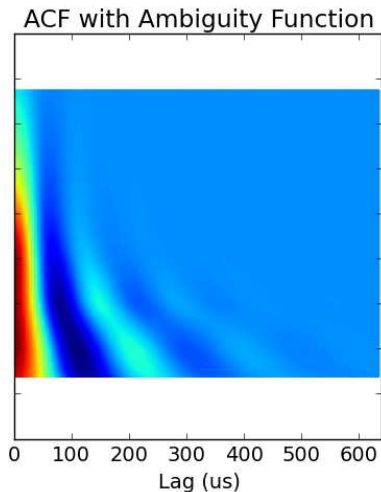
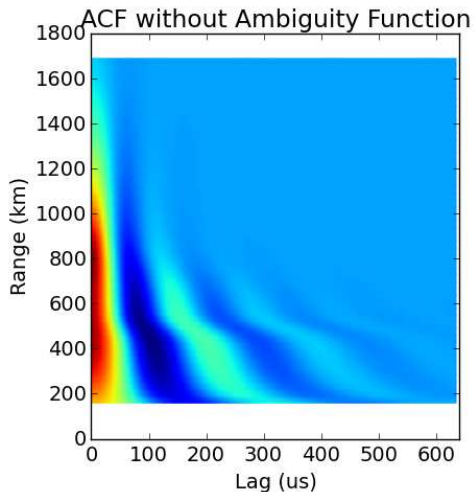


# Uncoded Long Pulse

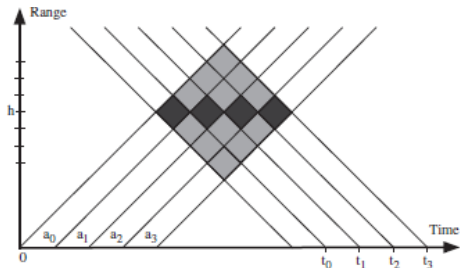


# Blurring of ACFs by Ambiguity Functions

A particular exaggerated example using 1.5 ms long pulses and a profile with a sharp  $T_e$  gradient at 500 km.



# Random Codes and Alternating Codes



$$a_0 a_1 v_0 v_1^* = a_0 \left( a_0 s_h^t + a_1 s_{h-1}^{t+\frac{1}{2}} + a_2 s_{h-2}^{t+1} + a_3 s_{h-3}^{t+\frac{3}{2}} \right) \times \\ a_1 \left( a_0 s_{h+1}^{t+\frac{1}{2}} + a_1 s_h^{t+1} + a_2 s_{h-1}^{t+\frac{3}{2}} + a_3 s_{h-2}^{t+2} \right)^*$$

$$E \{ a_0 a_1 v_0 v_1^* \} = E \{ s_h^t s_h^{*t+1} \} + a_0 a_2 E \left\{ s_{h-1}^{t+\frac{1}{2}} s_{h-1}^{*t+\frac{3}{2}} \right\} \\ + a_0 a_1 a_2 a_3 E \{ s_{h-2}^{t+1} s_{h-2}^{*t+2} \}$$

# Parameter Estimation and Inverse Problems

Given:

- Noisy measurements

$$\mathbf{Z} = \mathbf{Y} + \mathbf{W}$$

- The statistics of the noise

$$\text{Cov}\{\mathbf{W}\} = \mathbf{C}$$

- A forward model for what the noiseless data should be for a given set of parameters  $\beta$

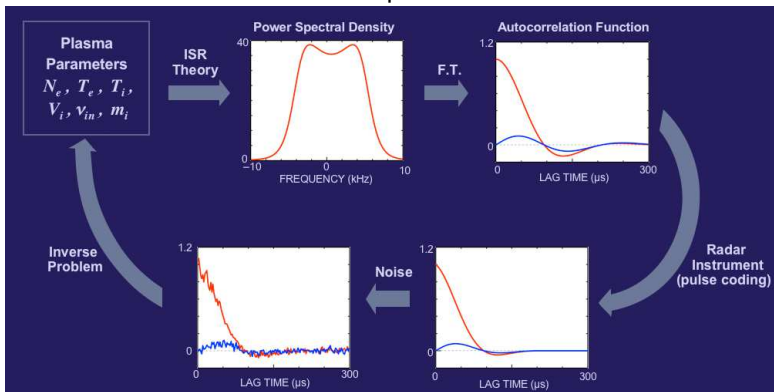
$$\mathbf{Y} = h(\beta)$$

How do I determine the best estimate of the parameters  $\beta$ ?

# Parameter Estimation Applied to ISR

For an ISR experiment:

- Noisy data are integrated lag-products (ACF estimators).
- Error properties determined from error analysis of estimators.  
Depends on SNR, self-clutter, etc.
- Parameters to be estimated are plasma state variables at each range:



# Creating Forward Models

The forward model has two portions

- 1 Physics and Chemistry (ISR Theory)
  - Assume Maxwellian distributions?
  - Constraints on  $T_e$  and  $T_i$ ?
  - Constraints on ion composition? Chemistry model?
  - Magnetic field effects
- 2 Instrumental Effects and Signal Processing
  - Sampling and Aliasing
  - Windowing
  - Ambiguity Functions

Best practice is to build the instrumental effects into the forward model.

**Do not manipulate the data in an attempt to undo the instrumental effects!**



# Least Squares Estimation

Least Squares Estimate:

$$\hat{\beta}_{\text{LS}} = \min_{\beta} [h(\beta) - \mathbf{Z}]^T C^{-1} [h(\beta) - \mathbf{Z}]$$

For diagonal  $C$

$$\hat{\beta}_{\text{LS}} = \min_{\beta} \sum_i \frac{[h_i(\beta) - Z_i]^2}{\sigma_i^2}$$

- If  $\mathbf{Z}$  is jointly gaussian, then the least-squares estimate is equivalent to the maximum likelihood estimate.
- A commonly used numerical technique for iteratively solving nonlinear least squares problems is the Levenberg-Marquardt algorithm
- Standard Levenberg-Marquardt packages:
  - FORTRAN: MINPACK lmdif.f and lmdcr.f
  - Python: `scipy.optimize.leastsq` (wrapper around `lmdif` and `lmdcr`)
  - Matlab: Optimization Toolbox `lsqnonlin`
  - IDL: LMFIT
- Levenberg-Marquardt requires a good initial guess

# Error Propagation (Linear Least Squares)

Linear Least Squares  $h(\beta) = H\beta$

$$\begin{aligned}\hat{\beta}_{\text{LS}} &= [H^T C^{-1} H]^{-1} H^T C^{-1} \mathbf{Z} \\ &= [\tilde{H}^T \tilde{H}]^{-1} \tilde{H}^T \tilde{\mathbf{Z}}\end{aligned}$$

where  $\tilde{H} = C^{-1/2} H$  and  $\tilde{\mathbf{Z}} = C^{-1/2} \mathbf{Z}$

Recall the property of jointly Gaussian random variables:

$$\mathbf{Y} = \mathbf{A}\mathbf{X} \Rightarrow \text{Cov}\{\mathbf{Y}\} = \mathbf{A}\text{Cov}\{\mathbf{X}\}\mathbf{A}^T$$

Thus

$$\begin{aligned}\text{Cov}\{\hat{\beta}_{\text{LS}}\} &= [\tilde{H}^T \tilde{H}]^{-1} \tilde{H}^T \text{Cov}\{\tilde{\mathbf{Z}}\} \tilde{H} [\tilde{H}^T \tilde{H}]^{-1} \\ &= [\tilde{H}^T \tilde{H}]^{-1}\end{aligned}$$

(Note  $\text{Cov}\{\tilde{\mathbf{Z}}\} = C^{-1/2} \text{Cov}\{\mathbf{Z}\} C^{-1/2} = I$ )

# Error Propagation (Nonlinear Least Squares)

Suppose we are minimizing

$$\hat{\beta}_{\text{LS}} = \min_{\beta} \sum_i \frac{[h_i(\beta) - Z_i]^2}{\sigma_i^2}$$

Linearize the problem in the vicinity of the final solution

$$\text{Cov} \{ \hat{\beta}_{\text{LS}} \} \approx [\tilde{J}^T \tilde{J}]^{-1}$$

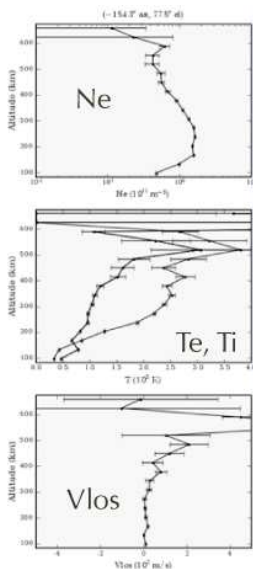
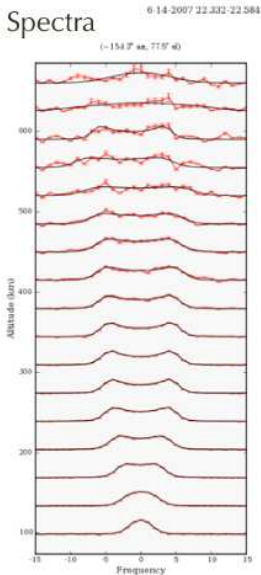
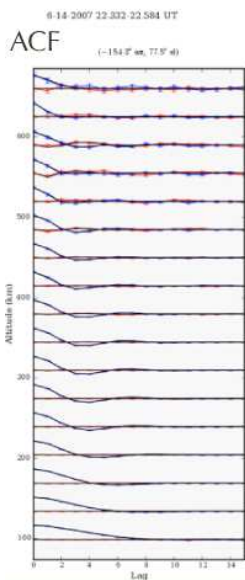
where the Jacobian  $\tilde{J}$  is evaluated at the final solution  $\beta = \hat{\beta}_{\text{LS}}$

$$\tilde{J} = \begin{pmatrix} \frac{1}{\sigma_0} \frac{\partial h_0}{\partial \beta_0} & \frac{1}{\sigma_0} \frac{\partial h_0}{\partial \beta_1} & \cdots & \frac{1}{\sigma_0} \frac{\partial h_0}{\partial \beta_{M-1}} \\ \frac{1}{\sigma_1} \frac{\partial h_1}{\partial \beta_0} & \frac{1}{\sigma_1} \frac{\partial h_1}{\partial \beta_1} & \cdots & \frac{1}{\sigma_1} \frac{\partial h_1}{\partial \beta_{M-1}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\sigma_{N-1}} \frac{\partial h_{N-1}}{\partial \beta_0} & \frac{1}{\sigma_{N-1}} \frac{\partial h_{N-1}}{\partial \beta_1} & \cdots & \frac{1}{\sigma_{N-1}} \frac{\partial h_{N-1}}{\partial \beta_{M-1}} \end{pmatrix}$$

$\tilde{J}$  is  $N \times M$  (tall and skinny)

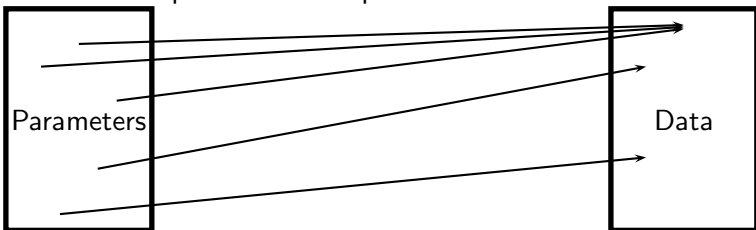
- Levenberg-Marquart computes  $\tilde{J}$  at every iteration internally
- Standard packages usually have an option to return either  $\tilde{J}$ , and/or  $[\tilde{J}^T \tilde{J}]^{-1}$  from the final iteration

## Example PFISR Long Pulse Fits



## Ill-Posed and Ill-Conditioned Problems

What happens if my forward model maps different points in parameter space to almost the same points in data space?



- **Ill-Posed Problem:** Multiple points in parameters space map to exactly the same point in data space.

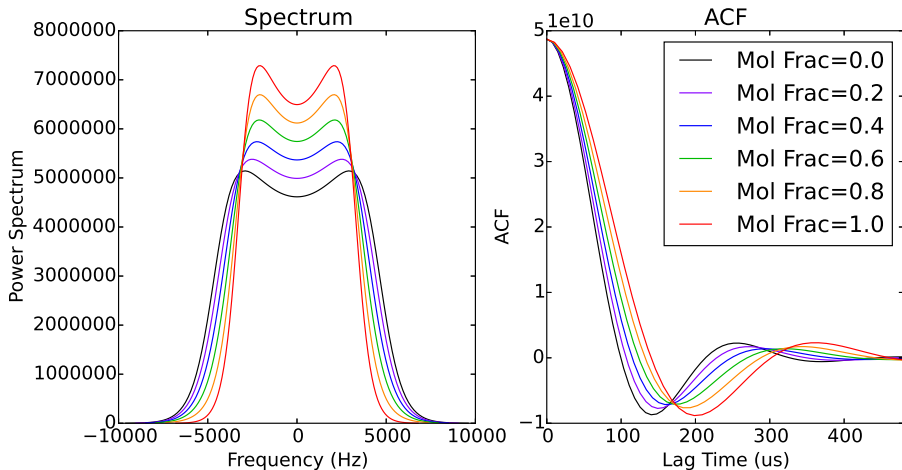
$\left[ \tilde{H}^T \tilde{H} \right]$  is singular, inverse problem is impossible

- **Ill-Conditioned Problem:** Multiple points in parameters space map to nearly the same point in data space.

$\left[ \tilde{H}^T \tilde{H} \right]$  is nearly singular, inverse problem is unstable given noisy data

# III-Conditioned ISR Theory: Molecular Ion Chemistry

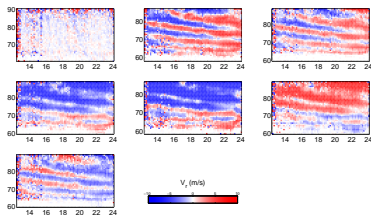
Mixtures of  $O^+$  and  $O_2^+$  using  $N_e = 10^{11}$ ,  $T_e = T_i = 1000$  K



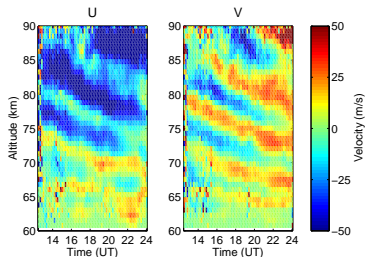
ISR spectrum measures  $\sqrt{\frac{T_i}{m_i}}$ , ambiguity between  $T_i$  and  $m_i$

# Mesospheric Vector Neutrals Winds

## Line of Sight Velocities



## Fitted Horizontal Velocities



$$\begin{pmatrix} V_{r,1} \\ \vdots \\ V_{r,7} \end{pmatrix} = \begin{pmatrix} \cos(\theta_1) \sin(\phi_1) & \cos(\theta_1) \sin(\phi_1) & \sin(\theta_1) \\ \vdots & \vdots & \vdots \\ \cos(\theta_7) \sin(\phi_7) & \cos(\theta_7) \sin(\phi_7) & \sin(\theta_7) \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$\mathbf{V}_r = \mathbf{D}\mathbf{U}$$

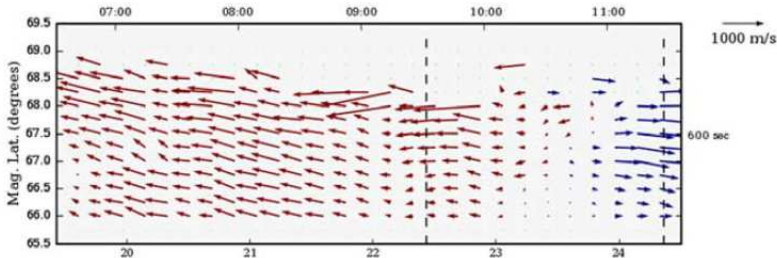
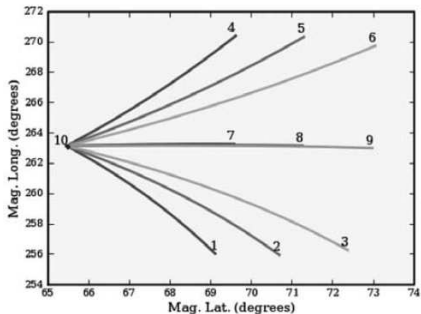
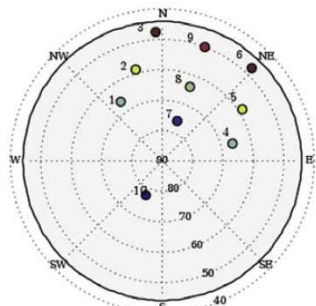
$$\mathbf{U} = (\mathbf{D}^T \mathbf{C}_{V_r}^{-1} \mathbf{D})^{-1} \mathbf{D}^T \mathbf{C}_{V_r}^{-1} \mathbf{V}_r$$

# F-region 1-D Vector Electric Fields

- In F-region assume  $\mathbf{v}_i = \frac{\mathbf{E} \times \mathbf{B}}{B^2}$
- Assume  $\mathbf{E} \cdot \mathbf{B} = 0$  (no parallel fields)
- LOS velocity is related to  $\mathbf{E}$  perpendicular to LOS and  $\mathbf{B}$
- Assume  $\mathbf{E}$  is uniform in magnetic longitude, but varies with magnetic latitude
- Assume  $\mathbf{E}$  fields map along equipotential field lines
- Different range gates correspond to different magnetic latitudes
- Fit for 2-components of  $\mathbf{E}$  as a function of magnetic latitude



# PFISR Electric Field Estimation



# Interpretation of E-region Ion Velocities

Ion Momentum Equation:

$$0 = e(\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) - m_i \nu_{in} (\mathbf{u}_i - \mathbf{u}_n)$$

Collisional Limit (D-region):  $\mathbf{u}_i = \mathbf{u}_n$

Collisionless Limit (F-region):  $\mathbf{u}_i = \frac{\mathbf{E} \times \mathbf{B}}{B^2}$

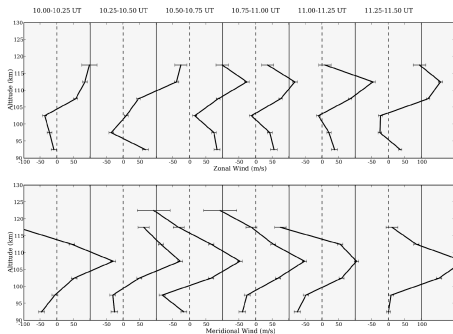
$$\text{E-region: } \mathbf{u}_i = \begin{pmatrix} \frac{1}{1+\kappa_i^2} & \frac{-\kappa_i}{1+\kappa_i^2} & 0 \\ \frac{\kappa_i}{1+\kappa_i^2} & \frac{1}{1+\kappa_i^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \left[ \mathbf{u}_n + \frac{e}{m_i \nu_{in}} \mathbf{E} \right]$$

$$\kappa_i \equiv \frac{eB}{m_i \nu_{in}}$$

# E-region Neutral Wind Estimation

- Estimate vector E-region ion velocities from E-region LOS velocity
- Estimate vector F-region electric fields from F-region LOS velocity
- Map electric fields from F-region to E-region along equipotential field lines
- Solve for  $\mathbf{u}_n$

$$\mathbf{u}_n = \mathbf{u}_i - \frac{e}{m_i \nu_{in}} (\mathbf{E} + \mathbf{u}_i \times \mathbf{B})$$



Heinselmann and Nicolls (2008) Radio Sci.

# Derived Electrodynamical Parameters

- Conductivity

$$\sigma_P = N_e e^2 \left( \frac{\nu_{en}/m_e}{\nu_{en}^2 + \Omega_e^2} + \frac{\nu_{in}/m_i}{\nu_{in}^2 + \Omega_i^2} \right)$$
$$\sigma_H = N_e e^2 \left( \frac{\Omega_e/m_e}{\nu_{en}^2 + \Omega_e^2} - \frac{\Omega_i/m_i}{\nu_{in}^2 + \Omega_i^2} \right)$$

- Horizontal Currents

$$\mathbf{J} = \sigma_P (\mathbf{E} + \mathbf{u}_n \times \mathbf{B}) - \sigma_H \left[ (\mathbf{E} + \mathbf{u}_n \times \mathbf{B}) \times \frac{\mathbf{B}}{B} \right]$$

- Joule Heating

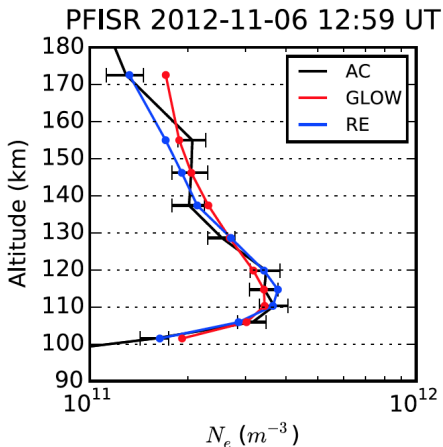
$$Q_J = \mathbf{J} \cdot (\mathbf{E} + \mathbf{u}_n \times \mathbf{B})$$
$$= \sigma_P |\mathbf{E} + \mathbf{u}_n \times \mathbf{B}|^2$$

See Thayer (1998) *JGR* and Thayer and Semeter (2004) *JASTP*

# Precipitation Characteristics from $N_e$ Profile Inversion

- Input  $N_e$  profiles vs altitude (up-B beam)
- Estimate precipitating energy flux and characteristic energy
- Use a forward model of energetic electron transport, impact ionization, and recombination (e.g. GLOW).

Kaeppler et al. (2015) *JGR*.



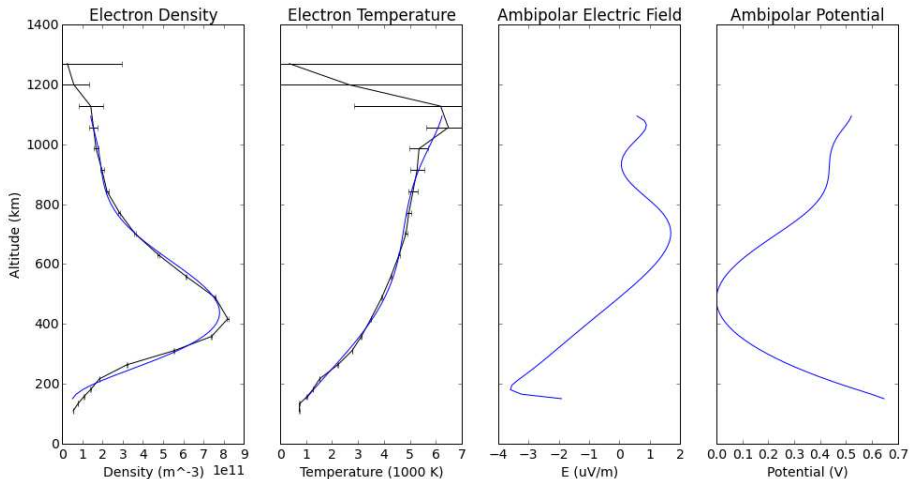
Best fit GLOW parameters:

$$Q_0 = 7.3 \pm 0.8 \text{ mW/m}^2,$$

$$E_0 = 5.0 \pm 0.2 \text{ keV}$$

# Ambipolar Electric Field (Sondrestrom up-B Example)

$$E_{\parallel} = -\frac{1}{eN_e} \nabla_{\parallel} (N_e k_B T_e)$$



## Ongoing Research Areas

Researchers are continuing to innovate and find new ways to extract more information from ISR data.

- Ion outflow fluxes
- Topside and plasmasphere parameters
- Neutral temperature derived from  $T_e$  and  $T_i$
- Neutral density derived from multi-frequency ISR
- Electron and ion heat fluxes
- Ion temperature anisotropy
- Non-Maxwellian distribution functions
- Gravity wave frequencies and wavevectors
- Energetic electron distributions from plasma line powers

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