Basic Radar 2: Fundamentals of Radar Signals

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Continuous Fourier Transform

$$\mathcal{F}\left\{f(t)\right\} = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt \qquad \mathcal{F}^{-1}\left\{F(\omega)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t}\, d\omega$$

Properties:

- Exists when $\int_{-\infty}^{\infty} |f(t)|^2 \, dt$ converges
- Linearity $\mathcal{F}\left\{\alpha f(t) + \beta g(t)\right\} = \alpha \mathcal{F}\left\{f(t)\right\} + \beta \mathcal{F}\left\{g(t)\right\}$
- Time reversal $\mathcal{F}\left\{f(-t)\right\} = \mathcal{F}\left\{f(t)\right\}^*$
- Time Delay $\mathcal{F}\left\{f(t - t_0)\right\} = e^{-j\omega t_0} \mathcal{F}\left\{f(t)\right\}$
- Convolution Theorem:

$$g(t) = f(t) * h(t) = \int_{-\infty}^{\infty} f(t') h(t - t')\, dt'$$

$$\mathcal{F}\left\{g(t)\right\} = \mathcal{F}\left\{f(t)\right\} \times \mathcal{F}\left\{h(t)\right\}$$

Duality Between Time and Frequency

Fourier transform is a unitary operator: $\mathcal{F}^{-1} = \frac{1}{2\pi}\mathcal{F}^*$.

All properties have an dual

- Frequency reversal $\mathcal{F}^{-1}\left\{F(-\omega)\right\} = \mathcal{F}^{-1}\left\{F(\omega)\right\}^*$
- Frequency shift $\mathcal{F}^{-1}\left\{F(\omega-\omega_0)\right\} = e^{j\omega_0 t}\mathcal{F}^{-1}\left\{F(\omega)\right\}$
- Dual Convolution Theorem:

$$G(\omega) = F(\omega) * H(\omega) = \int_{-\infty}^{\infty} F(\omega')H(\omega-\omega')\,d\omega'$$

$$\mathcal{F}^{-1}\left\{G(\omega)\right\} = 2\pi\mathcal{F}^{-1}\left\{F(\omega)\right\} \times \mathcal{F}^{-1}\left\{H(\omega)\right\}$$

Uncertainty principle (Gabor Limit):

$$\left(\int_{-\infty}^{\infty} t^2\left|f(t)\right|^2\,dt\right)\left(\int_{-\infty}^{\infty} \omega^2\left|F(\omega)\right|^2\,d\omega\right) \geq \frac{1}{16\pi^2}\left(\int_{-\infty}^{\infty}\left|f(t)\right|^2\,dt\right)^2$$

Narrow in time $\leftrightarrow$ Wide in frequency

Wide in time $\leftrightarrow$ Narrow in frequency

Discrete-Time Fourier Transform

Suppose I sample data with sampling period T such that $f_n = f(nT)$. The DTFT is defined as

$$\mathrm{DTFT}\left\{f\right\} = \sum_{n=-\infty}^{\infty} f_n e^{-j\omega nT}$$

$$\mathrm{DTFT}^{-1}\left\{F(\omega)\right\} = \frac{T}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} F(\omega)e^{j\omega nT}\,d\omega$$

- The DTFT maps a discrete sequence onto a function of continuous frequency
- The DTFT is $\frac{2\pi}{T}$-periodic in ω since $e^{-j\omega nT} = e^{-j\left(\omega+\frac{2\pi}{T}\right)nT}$.
- The DTFT is related to the CFT by:

$$\mathrm{DTFT}\left\{f\right\} = \mathcal{F}\left\{\sum_{-\infty}^{\infty} f(nT)\delta\left(t-nT\right)\right\}$$

Discrete Fourier Transform

Given N discrete samples:

$$F_k = \sum_{n=0}^{N-1} f_n e^{-2\pi j \frac{nk}{N}} \quad k \in [0, N-1] \qquad \text{Forward DFT}$$

$$f_n = \frac{1}{N} \sum_{k=0}^{N-1} F_k e^{2\pi j \frac{nk}{N}} \quad n \in [0, N-1] \qquad \text{Inverse DFT}$$

- The Fast Fourier Transform (FFT) is an algorithm for computing the DFT that is $\mathcal{O}(N \log N)$
- The DFT is a discrete sampling of the DTFT.
- If $f_n = f(nT)$ for sampling period T, then the corresponding set of discrete frequencies are

$$\omega_k = 2\pi \frac{k}{NT}$$

- Frequency resolution is $\Delta\omega = \frac{2\pi}{NT}$

Sampling and Aliasing

Periodicity of complex exponentials:

$$e^{2\pi j \frac{nk}{N}} = e^{2\pi j \frac{nk}{N} + 2\pi j m} \quad \forall m \in \mathbb{Z}$$

From this it follows that

$$F_k = \sum_{n=0}^{N-1} f_n e^{-2\pi j \frac{nTk}{NT}} = \sum_{n=0}^{N-1} f_n e^{-2\pi j \frac{nTk + mnTN}{NT}}$$

$$\omega_k = 2\pi \left(\frac{k}{NT} + \frac{m}{T} \right)$$

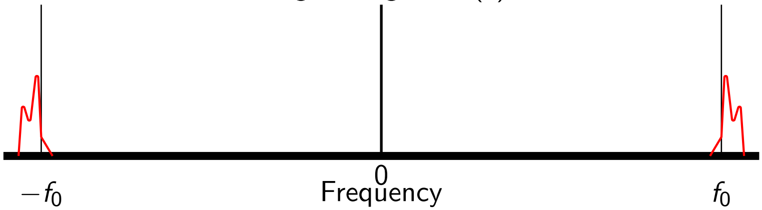
For example, with $T = 2 \text{ ms}$, $\frac{1}{T} = 500 \text{ Hz}$:

$$\cdots \leftrightarrow -925 \text{ Hz} \leftrightarrow -425 \text{ Hz} \leftrightarrow 75 \text{ Hz} \leftrightarrow 575 \text{ Hz} \leftrightarrow 1075 \text{ Hz} \leftrightarrow \cdots$$

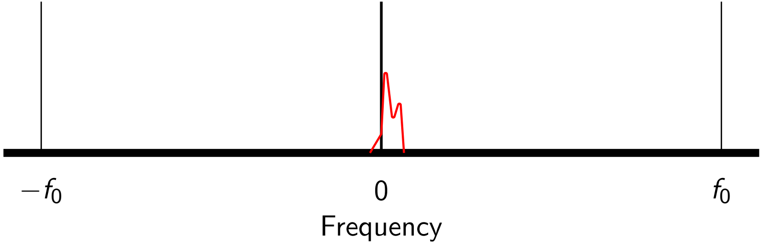
- The DFT returns frequencies from 0 to $\frac{1}{T}$ by default. A common convention is to report the spectrum from $-\frac{1}{2T}$ to $\frac{1}{2T}$ (fftshift).
- **Nyquist Sampling Theorem:** All signal frequencies must be $|\omega| < 2\pi\frac{1}{2T}$ for the DFT and DTFT to avoid frequency aliasing.

Baseband Sampling

Original Signal: $v(t)$



Baseband Signal: $v_n \left\langle e^{-2\pi j f_0 n T} v(nT) \right\rangle$



Windowing and Frequency Resolution

What are the ramifications of using a finite length segement of data in the DFT?

$$F_k = \sum_{n=0}^{N-1} f_n e^{-j\omega_k nT} \quad \text{DFT}$$

$$= \sum_{n=-\infty}^{\infty} f_n r_n^N e^{-j\omega_k nT} \quad \text{DTFT}$$

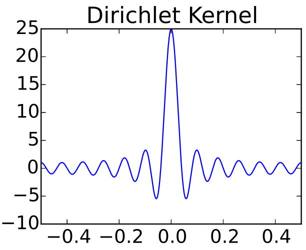
where the length N rectangle function is:

$$r_n^N = \begin{cases} 1 & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$$

By the convolution theorem for the DTFT:

$$\mathrm{DTFT}\left\{f \times r^N\right\} = \mathrm{DTFT}\left\{f\right\} * \mathrm{DTFT}\left\{r^N\right\}$$

$$\mathrm{DTFT}\left\{r^N\right\} = e^{-j\frac{N-1}{2}\omega T} \frac{\sin\left(\frac{N}{2}\omega T\right)}{\sin\left(\frac{1}{2}\omega T\right)}$$



Width of Dirichlet kernel is related to NT.

Zero-padded Fourier Transforms

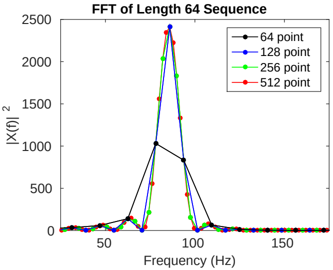
What happens if I try to sample a finer grid of frequencies than the standard DFT?

$$F'_k = \sum_{n=0}^{N-1} f_n e^{-2\pi j \frac{nk}{K}} \quad k \in [0, K-1], K > N$$

This is equivalent to zero padding the original sequence, then taking a standard DFT.

$$f'_n = \begin{cases} f_n & n < N \\ 0 & N < n < K \end{cases}$$

$$F'_k = \sum_{n=0}^{K-1} f'_n e^{-2\pi j \frac{nk}{K}} \quad k \in [0, K-1]$$

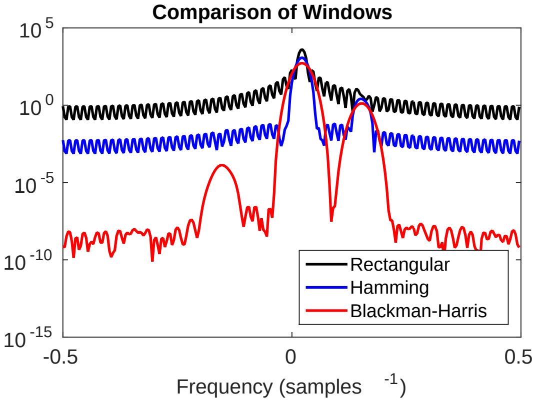


The length N rectangular window sets the real frequency resolution. Zero padding just interpolates the windowed spectrum.

Effects of Other Window Functions

Chose tapered window weights w_n to combat effects of rectangular window

$$F_k = \sum_{n=0}^{N-1} w_n f_n e^{-j\omega_k nT}$$



Digital Filters

Finite Impulse Response Filter (FIR)

$$y_k = \sum_{n=0}^{N-1} h_n x_{k-n}$$

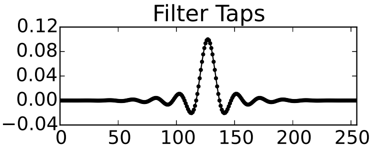
Infinite Impulse Response Filter (IIR)

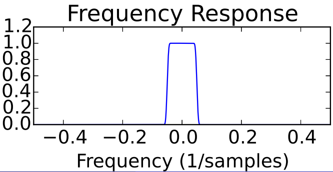
$$y_k = \sum_{n=0}^{N-1} h_n x_{k-n} + \sum_{m=1}^{M-1} g_m y_{k-m}$$

- FIR used extensively in radar signal processing.
- Impulse response is the output when $x_n = \delta_n$. For FIR filters this is h_n.
- Frequency response of an FIR filter:

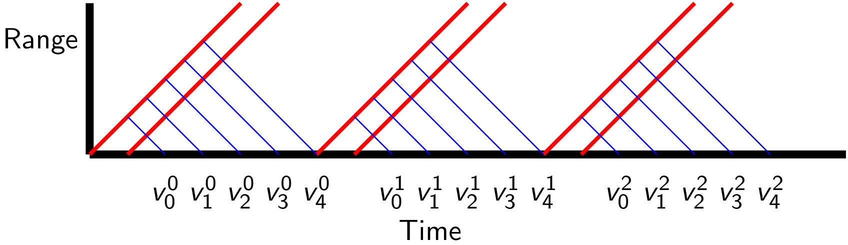
$$H(\omega) = \sum_{n=0}^{N-1} h_n e^{-j\omega n T} \quad \text{DTFT of Impulse Response}$$

Example: 255-tap FIR Blackman filter used at PFISR





Range-Time Diagram



Interpret the 1-D receiver voltage as a function of time as if it were a function of both range and time.

- v_n^k is the sample of range gate n after pulse k
- n is the "fast-time" dimension (range), and k is the "slow-time" dimension (time)
- Range gate n is at a range of $R = \frac{ct_n}{2}$
- Each slow-time sample is separated by one interpulse period (IPP)

Range-Doppler Processing

Range-Time Data Array:

$$\begin{pmatrix} v_0^0 & v_0^1 & \cdots & v_0^{K-1} \\ v_1^0 & v_1^1 & \cdots & v_1^{K-1} \\ \vdots & \vdots & \vdots & \vdots \\ v_{N-1}^0 & v_{N-1}^1 & \cdots & v_{N-1}^{K-1} \end{pmatrix}$$

Take an FFT along the slow-time dimension for each range

$$u_n^m = \sum_{k=0}^{K-1} v_n^k e^{2\pi j \frac{mk}{K}}$$

Range-Frequency Data Array:

$$\begin{pmatrix} u_0^0 & u_0^1 & \cdots & u_0^{K-1} \\ u_1^0 & u_1^1 & \cdots & u_1^{K-1} \\ \vdots & \vdots & \vdots & \vdots \\ u_{N-1}^0 & u_{N-1}^1 & \cdots & u_{N-1}^{K-1} \end{pmatrix}$$

Physically $|u_n^m|^2$ is proportional to the received power at

- Range $r_n = \frac{ct_n}{2}$
- Frequency $\omega_m = 2\pi \frac{m}{K\tau_{\mathrm{IPP}}}$
- Dopper velocity $V_m = \frac{c}{2}\frac{\omega_m}{2\pi f_0}$

Doppler Aliasing

The FFTs are performed along the slow time dimension, hence the effective sampling rate is τ_{IPP}.

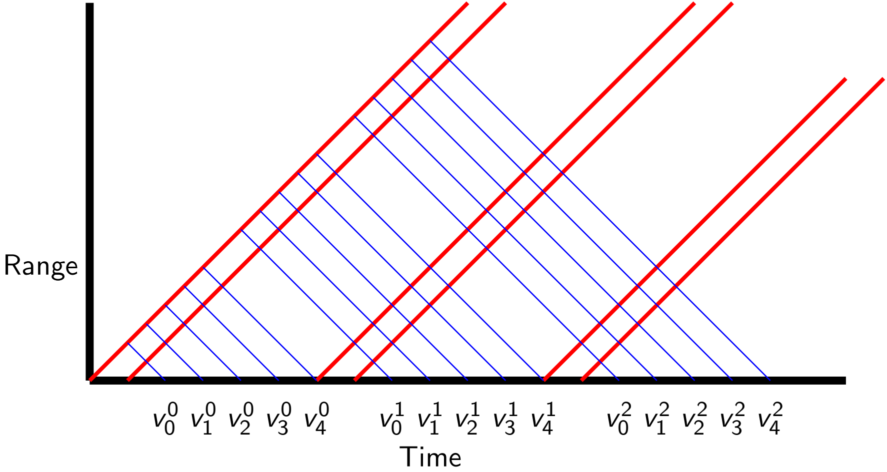
In order to avoid aliasing in the FFTs for each range:

$$\tau_{\mathrm{IPP}} < \frac{1}{2f_{\mathrm{max}}}$$

Short IPPs implies pulses very close together.

Is there any problem with having pulses very close together?

Range Aliasing



$$r_n = \frac{ct_n}{2} + m\frac{c\tau_{\mathrm{IPP}}}{2} \qquad \text{for any integer } m$$

Underspread and Overspread Targets

Shorter IPP:

- Captures wide bandwidth
- Leads to range aliasing

Longer IPP:

- Captures a large range extent
- Leads to frequency aliasing

If an IPP exists than can avoid both range and frequency aliasing the target is **underspread**. Otherwise it is **overspread**.

Underspread Example: D-region ISR at 450 MHz

- Ranges 0-90 km, Frequencies $\pm$75 Hz
- 2 ms IPP gives 300 km unambiguous range, $\pm$250 Hz frequencies

Overspread Example: F-region ISR at 450 MHz

- Ranges 0-600 km, Frequencies $\pm$10 kHz
- 600 km unambiguous range requires a $> 4\ \mathrm{ms}$ IPP
- $\pm$10 kHz unambiguous frequencies requires a $< 50\ \mu\mathrm{s}$ IPP

Gaussian Random Variables

A Gaussian random variable X has the following probability density function (Normal Distribution):

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{x-\mu}{2\sigma^2}\right\}$$

$$E\{X\} = \mu$$

$$Var\{X\} = E\left\{(X-\mu)^2\right\} = E\left\{X^2\right\} - \mu^2 = \sigma^2$$

A vector of random variables $\mathbf{X} = [X_0 X_1 X_2 \cdots X_{N-1}]^T$ is jointly Gaussian if each in they have the joint PDF

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{N}{2}} |C|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}[\mathbf{x}-\mu]^T C^{-1} [\mathbf{x}-\mu]\right\}$$

$$E\{\mathbf{X}\} = \mu$$

$$Cov\{\mathbf{X}\} = E\left\{[\mathbf{X}-\mu][\mathbf{X}-\mu]^T\right\} = C$$

Properties of Jointly Gaussian Random Variables

- Linear combinations:

$$Z = \alpha X + \beta Y + \gamma \quad E\{Z\} = \alpha E\{X\} + \beta E\{Y\} + \gamma$$
$$Var\{Z\} = \alpha^2 Var\{X\} + \beta^2 Var\{Y\} + 2\alpha\beta Cov\{X, Y\}$$

- Matrix generalization:
  $\mathbf{Y} = \mathbf{AX} + \mathbf{b} \quad E\{\mathbf{Y}\} = \mathbf{AX} + \mathbf{b} \quad Cov\{\mathbf{Y}\} = \mathbf{A} Cov\{\mathbf{X}\}\mathbf{A}^T$

Special cases for zero mean random variables:

- Odd moments are zero:
  $E\{V_1\} = E\{V_1 V_2 V_3\} = E\{V_1 V_2 V_3 V_4 V_5\} = \cdots = 0$
- Fourth moment theorem: $E\{V_1 V_2 V_3 V_4\} =$
  $E\{V_1 V_2\} E\{V_3 V_4\} + E\{V_1 V_3\} E\{V_2 V_4\} + E\{V_1 V_4\} E\{V_2 V_3\}$
- General even moment theorem (Isserlis' Theorem)
  $E\{V_1 V_2 \cdots V_{2n-1} V_{2n}\} = \sum \prod E\{V_i V_j\}$

Complex Gaussian Random Variables

ISR signals are complex valued, zero mean, and random phase.

$$V = V_R + jV_I \qquad E\{V_R\} = E\{V_I\} = 0$$

$$E\{VV^*\} = \sigma^2 \qquad E\{V_R V_I\} = 0 \qquad Cov\left\{\begin{pmatrix} V_R \\ V_I \end{pmatrix}\right\} = \frac{1}{2}\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}$$

When we talk about correlations between ISR signals

$$E\{V_1 V_1^*\} = \sigma_1^2 \quad E\{V_2 V_2^*\} = \sigma_2^2$$
$$E\{V_1 V_2^*\} = \rho = \rho_R + j\rho_I$$

What we really mean is

$$V_1 = V_{1R} + jV_{1I} \qquad V_2 = V_{2R} + jV_{2I}$$

$$Cov\left\{\begin{pmatrix} V_{1R} \\ V_{1I} \\ V_{2R} \\ V_{2I} \end{pmatrix}\right\} = \frac{1}{2}\begin{pmatrix} \sigma_1^2 & 0 & \rho_R & -\rho_I \\ 0 & \sigma_1^2 & \rho_I & \rho_R \\ \rho_R & \rho_I & \sigma_2^2 & 0 \\ -\rho_i & \rho_R & 0 & \sigma_2^2 \end{pmatrix}$$

Stochastic Processes: Definitions and Terminology

- Stochastic Process (aka Random Process): $V(t)$ where value at every time is a random variable
- Gaussian Stochastic Process:
  - PDF of each $V(t)$ is a Gaussian distribution (aka normal distribution)
  - Joint PDF of any subset of samples of $V(t)$ is a jointly Gaussian distribution (aka Multivariate Normal Distribution)
- Moments of a Stochastic Process:
  - Mean: $\bar{V}(t) = E\left\{V(t)\right\}$
  - Autocorrelation: $R\left(t, t-\tau\right) = E\left\{V(t)V^*(t-\tau)\right\}$
  - Autocovariance:
    $C\left(t, t-\tau\right) = E\left\{\left[V(t) - \bar{V}(t)\right]\left[V^*(t-\tau) - \bar{V}^*(t-\tau)\right]\right\} = R\left(t, t-\tau\right) - \bar{V}(t)\bar{V}^*(t-\tau)$
- (Wide Sense) Stationary Stochastic Process
  - $\bar{V}(t) = \bar{V}$ is independent of t
  - $R(t, t-\tau) = R(\tau)$ is independent of t
- ISR signals are Gaussian, zero mean, and stationary as long as the ionospheric state parameters are constant.

Examples of Discrete Stochastic Process

Gaussian white noise W_n:

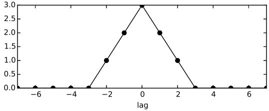
$$\bar{W} = 0 \qquad R_\ell = E\left\{W_n W_{n-\ell}^*\right\} = \begin{cases} \sigma_0^2 & \ell = 0 \\ 0 & \ell \neq 0 \end{cases}$$

3-point running sum of Gaussian white noise

$$V_n = W_n + W_{n-1} + W_{n-2}$$

$$R_\ell = E\left\{\left[W_n + W_{n-1} + W_{n-2}\right]\left[W_{n-\ell} + W_{n-\ell-1} + W_{n-\ell-2}\right]^*\right\}$$

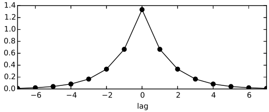
$$= \begin{cases} (3 - |\ell|)\,\sigma_0^2 & |\ell| < 3 \\ 0 & |\ell| \geq 3 \end{cases}$$



Autoregressive model

$$Y_n = \alpha Y_{n-1} + W_n \qquad Y_n = W_n + \alpha W_{n-1} + \alpha^2 W_{n-2} + \alpha^3 W_{n-3} + \cdots$$

$$R_\ell = \frac{\alpha^{|\ell|}}{1 - \alpha^2}\sigma_0^2$$



Examples of Continuous Random Processes

Voltage across a warm resistor (Nyquist Noise Theorem)

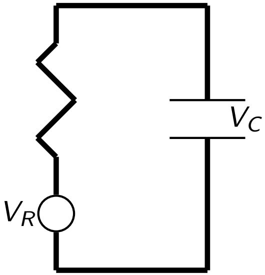
$$E\left\{V_R(t)\right\} = 0 \qquad E\left\{V_R(t)V_R^*(t-\tau)\right\} = 4Rk_B T\delta(\tau)$$

Voltage across capacitor connected to a warm resistor

$$V_C(t) = \frac{1}{RC}\int_{-\infty}^{t} V_R(t') - V_C(t')\, dt'$$

$$V_C(t) = \frac{1}{RC}\int_{-\infty}^{t} e^{\frac{t'-t}{RC}} V_R(t')\, dt'$$

$$E\left\{V_C(t)\ V_C(t-\tau)\right\} = \frac{2Rk_B T}{RC} e^{-\frac{|\tau|}{RC}}$$



Power Spectrum

- Fourier transform of a WSS random process does not exist since $\int_{-\infty}^{\infty} |V(t)|^2 \, dt \to \infty$
- Fourier transform of the ACF does exist, and is called the **Power Spectrum**.

$$S(\omega) \equiv \mathcal{F}\{R(\tau)\} = \int_{-\infty}^{\infty} R(\tau)\, e^{-j\omega\tau}\, d\tau$$

Properties of Power Spectrum:

- $S(\omega)$ is real and $S(\omega) \geq 0$ [follows from $R(\tau) = R(-\tau)^*$]
- $\int_{-\infty}^{\infty} S(\omega)\, d\omega = R(0)$ [total power]
- $\int_{\omega_1}^{\omega_2} S(\omega)\, d\omega =$ power in the band from ω_1 to ω_2
- If $y(t) = h(t) * x(t) \to S_Y(\omega) = |H(\omega)|^2\, S_X(\omega)$ where $H(\omega) = \mathcal{F}\{h(t)\}$
- Short correlation times $\leftrightarrow$ wide bandwidth and vice versa.

Power Estimation

Given K samples v_i of a WSS random process $V(t)$ with true power $P = R(0) = E\{V(t)V^*(t)\}$, and assuming the samples are far enough apart that they are uncorrelated

Power estimator:

$$\hat{P} = \frac{1}{K}\sum_{i=0}^{K-1} v_i v_i^*$$

Expected value of power estimator:

$$E\left\{\hat{P}\right\} = P \qquad \text{unbiased estimator}$$

Variance of power estimator:

$$\delta\hat{P}^2 = E\left\{\left(\hat{P} - P\right)^2\right\} = \frac{1}{K}P^2$$

Relative error $\frac{\delta\hat{P}}{P} = \frac{1}{\sqrt{K}}$

Signal Power Estimation with Added Noise

Given K samples $v_i = s_i + n_i$, and an independently known noise power, N

$$\hat{S} = \frac{1}{K}\sum_{i=0}^{K-1} v_i v_i^* - N$$

$$E\left\{\hat{S}\right\} = S \qquad \text{unbiased estimator}$$

$$Var\left\{\hat{S}\right\} = \frac{1}{K}(S+N)^2$$

$$\frac{\delta\hat{S}}{S} = \frac{1}{\sqrt{K}}\left(1 + \frac{1}{S/N}\right)$$

For example, $\frac{\delta\hat{S}}{S} = 0.5$ with a $S/N = 0.1$ requires $K = 484$.

ACF Estimation (Pulse-to-Pulse)

Now assume pulses are taken close together and are correlated.
Unbiased Estimator:

$$\hat{R}_\ell = \frac{1}{K-\ell} \sum_{n=\ell}^{K-1} v_n v_{n-\ell}^*$$

$$E\left\{\hat{R}_\ell\right\} = R_\ell$$

Biased Estimator:

$$\tilde{R}_\ell = \frac{1}{K} \sum_{n=\ell}^{K-1} v_n v_{n-\ell}^*$$

$$E\left\{\tilde{R}_\ell\right\} = \frac{K-\ell}{K} R_\ell \qquad \text{[triangular window]}$$

$$Var\left\{\tilde{R}_\ell\right\} = \frac{1}{K^2} \sum_{n=\ell}^{K-1} \sum_{m=\ell}^{K-1} \left|R_{m-n}\right|^2 \approx \frac{1}{K} \left|R_0\right|^2$$

Spectral Estimation (Pulse-to-Pulse Periodograms)

Given $v_0, v_1, v_2, ..., v_{K-1}$ I could compute

$$\check{V}_n = \sum_{k=0}^{K-1} v_k e^{-2\pi j \frac{nk}{K}} \quad n \in [0, K-1]$$

$$\check{S}_n = \left| \check{V}_n \right|^2$$

$$\check{R}_k = \frac{1}{K} \sum_{n=0}^{K-1} \check{S}_n e^{2\pi j \frac{nk}{K}}$$

This turns out to be biased by periodic wrap-around effects. For example:

$$K \check{R}_2 = v_2 v_0^* + v_3 v_1^* + \cdots + v_{K-1} v_{K-3}^* + v_0 v_{K-2}^* + v_1 v_{K-1}^*$$

This is called a periodogram, and generally shouldn't be used.

Zero-padded Periodograms

A better estimator is the zero padded periodogram:

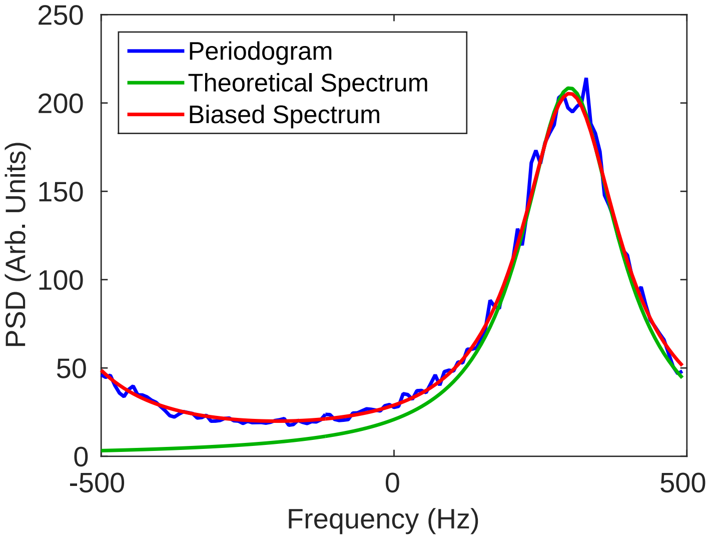
$$\tilde{V}_n = \sum_{k=0}^{K-1} v_k e^{-2\pi j \frac{nk}{2K}} \quad n \in [0, 2K-1] \quad \tilde{S}_n = \left|\tilde{V}_n\right|^2$$

$$\begin{aligned}\tilde{R}_k &= \frac{1}{2K} \sum_{n=0}^{2K-1} \tilde{S}_n e^{2\pi j \frac{nk}{2K}} \\ &= \frac{1}{K} \sum_{n=k}^{K-1} v_n v_{n-k}^*\end{aligned}$$

This provides a fast way to compute the biased ACF estimator using FFTs.

- $\tilde{R}_k$ is a sampled and triangularly windowed estimate of $R(\tau)$.
- $\tilde{S}_n$ is an aliased and smoothed estimate of $S(\omega)$.

Effects of Aliasing and Windowing on Periodograms



Combining Coherent and Incoherent Integration

Divide a sequence of M pulses into L sets of K such that $M = LK$.

$$\tilde{R}_\ell = \frac{1}{L}\sum_{\ell=0}^{L-1}\frac{1}{K}\sum_{k=\ell}^{K-1} v_k v_{k-\ell}^* \quad \ell \in [0, K-1]$$

Increasing the coherent integration time K gives

- Longer lags in $\tilde{R}$
- Higher frequency resolution in $\tilde{S}$

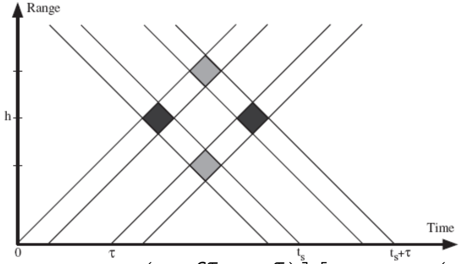
Increasing the incoherent integration time L gives

- Better statistics

In general K should be long enough to capture the correlation time of the process; longer is computationally wasteful.

Unlike the coherent integration samples, the incoherent integration intervals do not need to be contiguous in time.

Double Pulse Experiment



$$\begin{aligned}v(t_s+\tau)v^*(t_s) &= \left[s\left(h;t+\tau\right)+s\left(h+\frac{c\tau}{2};t+\frac{\tau}{2}\right)\right]\left[s\left(h;t\right)+s\left(h-\frac{c\tau}{2};t+\frac{\tau}{2}\right)\right] \\ &= s\left(h;t+\tau\right)s^*\left(h;t\right)+s\left(h;t+\tau\right)s^*\left(h-\frac{c\tau}{2};t+\frac{\tau}{2}\right) \\ &\quad + s\left(h+\frac{c\tau}{2};t+\frac{\tau}{2}\right)s^*\left(h;t\right)+s\left(h+\frac{c\tau}{2};t+\frac{\tau}{2}\right)s^*\left(h-\frac{c\tau}{2};t+\frac{\tau}{2}\right)\end{aligned}$$

ISR signals from disjoint altitudes are uncorrelated:

$$E\left\{s\left(h_1;t+\tau\right)s^*\left(h_2;t\right)\right\} = \begin{cases} R(h_1;\tau) & h_1 = h_2 \\ 0 & h_1 \neq h_2 \end{cases}$$

$$E\left\{v(t_s+\tau)v^*(t_s)\right\} = E\left\{s\left(h;t+\tau\right)s^*\left(h;t\right)\right\} = R(h;\tau)$$

Error Analysis of Double Pulse Experiment

Without self-clutter (dual polarization):

$$v_1 = s\left(h; t+\tau\right) + n\left(t+\tau\right) \qquad v_2 = s\left(h;t\right) + n(t)$$

$$\hat{R}(\tau) = \frac{1}{K}\sum_{i=0}^{K-1} v_{i1} v_{i2}^* \qquad E\left\{\hat{R}(\tau)\right\} = E\left\{v_1 v_2^*\right\} = R(h;\tau)$$

$$\begin{aligned}
Var\left\{\hat{R}(\tau)\right\} &= \frac{1}{K}\left[E\left\{v_1 v_2^* v_1^* v_2\right\} - \left|E\left\{v_1 v_2^*\right\}\right|^2\right] \\
&= \frac{1}{K}\left[E\left\{v_1 v_2^*\right\} E\left\{v_1^* v_2\right\} + E\left\{v_1 v_1^*\right\} E\left\{v_2^* v_2\right\}\right. \\
&\qquad \left. + \cancel{E\left\{v_1 v_2\right\} E\left\{v_2^* v_1^*\right\}} - \left|E\left\{v_1 v_2^*\right\}\right|^2\right] \\
&= \frac{1}{K} E\left\{v_1 v_1^*\right\} E\left\{v_2^* v_2\right\} \\
&= \frac{1}{K}\left[S(h) + N\right]^2 = \frac{S^2(h)}{K}\left[1 + \frac{1}{S(h)/N}\right]^2
\end{aligned}$$

Double Pulse Experiment with Self-Clutter

With self-clutter (single polarization):

$$v_1 = s\left(h; t+\tau\right) + s\left(h + \frac{c\tau}{2}; t + \frac{\tau}{2}\right) + n\left(t+\tau\right)$$

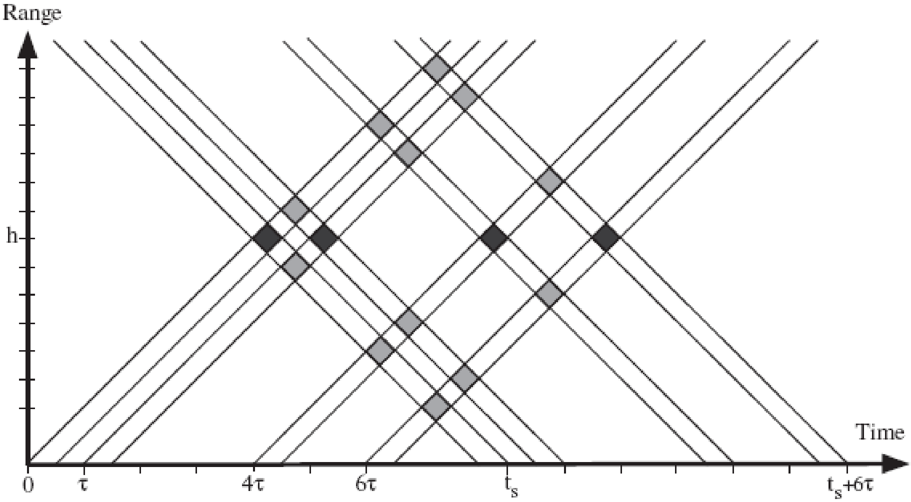
$$v_2 = s\left(h; t\right) + s\left(h - \frac{c\tau}{2}; t + \frac{\tau}{2}\right) + n(t)$$

$$\hat{R}(\tau) = \frac{1}{K}\sum_{i=0}^{K-1} v_{i1} v_{i2}^* \qquad E\left\{\hat{R}(\tau)\right\} = E\left\{v_1 v_2^*\right\} = R(h;\tau)$$

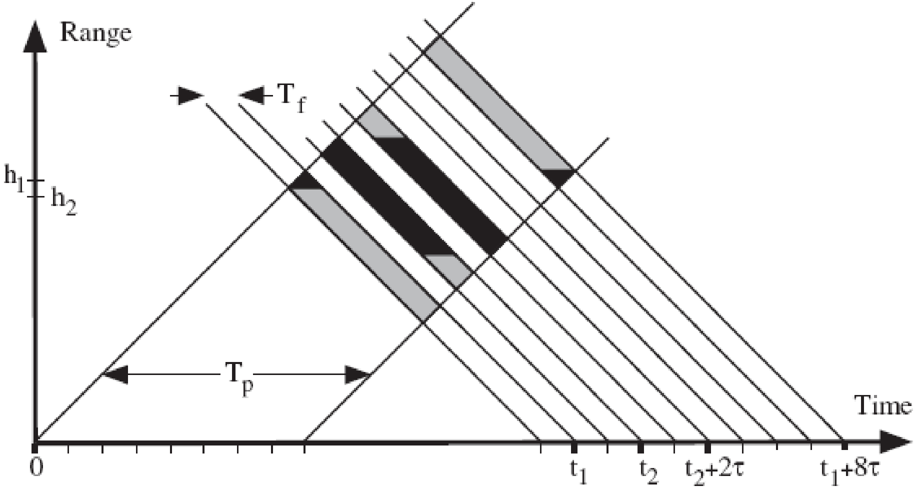
$$\begin{aligned} Var\left\{\hat{R}(\tau)\right\} &= \frac{1}{K} E\left\{v_1 v_1^*\right\} E\left\{v_2^* v_2\right\} \\ &= \frac{1}{K}\left[S(h) + S\left(h + \frac{c\tau}{2}\right) + N\right]\left[S(h) + S\left(h - \frac{c\tau}{2}\right) + N\right] \end{aligned}$$

Signal-to-Noise ratio $\rightarrow$ Signal-to-(Noise+Clutter) ratio

Multipulse Experiments



Uncoded Long Pulse



Amplitude Ambiguity Function

After transmitting a pulse envelope $s(t)$, the scattered signal is

$$x(t) = \int d^3\mathbf{r}\, e^{j\mathbf{k}\cdot r} s\left(t - \frac{2r}{c}\right) \Delta N_e \left(\mathbf{r}, t - \frac{r}{c}\right)$$

The receiver records a filtered and sampled version of the scattered signal

$$\begin{aligned} y\left(t_s\right) &= \int dt\, x(t) h^*\left(t_s - t\right) \\ &= \int dt d^3\mathbf{r}\, e^{j\mathbf{k}\cdot\mathbf{r}} s\left(t - \frac{2r}{c}\right) \Delta N_e \left(\mathbf{r}, t - \frac{r}{c}\right) h^*\left(t_s - t\right) \end{aligned}$$

Define the **amplitude ambiguity function**

$$W_{t_s} \equiv s\left(t - \frac{2r}{c}\right) h^*\left(t_s - t\right)$$

$$y\left(t_s\right) = \int dt d^3\mathbf{r}\, e^{j\mathbf{k}\cdot\mathbf{r}} W_{t_s}(t, r) \Delta N_e \left(\mathbf{r}, t - \frac{r}{c}\right)$$

Range-Lag Ambiguity Function

When we form ACFs, we take products of samples and average:

$$\langle y(t_{s2})\, y^*(t_{s1})\rangle = \int dt_1 dt_2 d^3\mathbf{r}_1 d^3\mathbf{r}_2 e^{j\mathbf{k}\cdot(\mathbf{r}_2-\mathbf{r}_1)} \left\langle \Delta N_e\left(\mathbf{r}_2, t_2 - \frac{r_2}{c}\right) \Delta N_e^*\left(\mathbf{r}_1, t_1 - \frac{r_1}{c}\right)\right\rangle W_{ts2}(t_2, r_2)\, W_{ts1}^*(t_1, r_1)$$

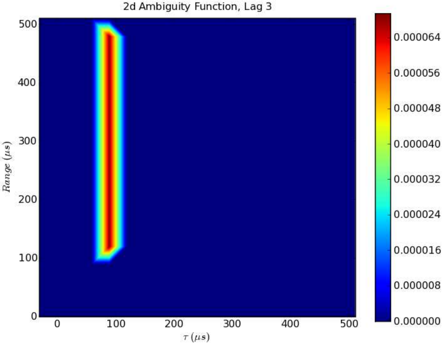
Change variables $t_1 = t \quad t_2 = t + \tau \quad \mathbf{r}_1 = \mathbf{r} \quad \mathbf{r}_2 = \mathbf{r} + \mathbf{r}'$
Perform $\mathbf{r}'$ integral and take expected value

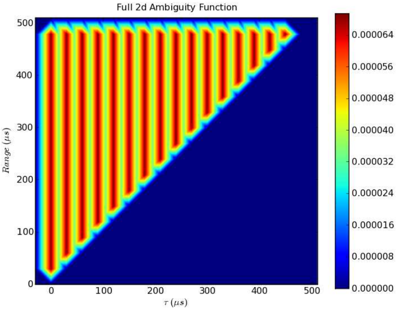
$$\langle y(t_{s2})\, y^*(t_{s1})\rangle = \int d\tau d^3\mathbf{r}\, R(\mathbf{k}, \tau, \mathbf{r}) \underbrace{\int dt\, W_{ts2}(t+\tau, r)\, W_{ts1}^*(t, r)}_{W_{t_{s1},t_{s2}}(\tau, r)}$$

The measured lag-product is the ISR ACF we want $R(\mathbf{k}, \tau, \mathbf{r})$ blurred the the **range-lag ambiguity function** $W_{t_{s1},t_{s2}}(\tau, r)$

2-D Range-lag Ambiguity Function of Long Pulse

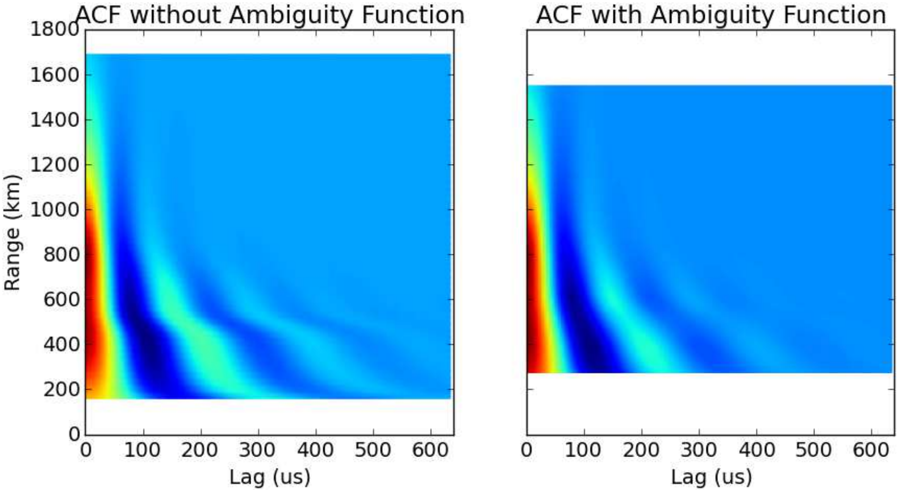
Ambiguity function with a boxcar filter. 480 μs long pulse, 30 μs sampling.



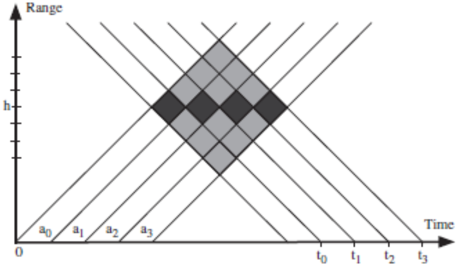


Theoretical Long Pulse Examples

A particular exaggerated example using 1.5 ms long pulses and a profile with a sharp T_e gradient at 500 km.



Random Codes and Alternating Codes

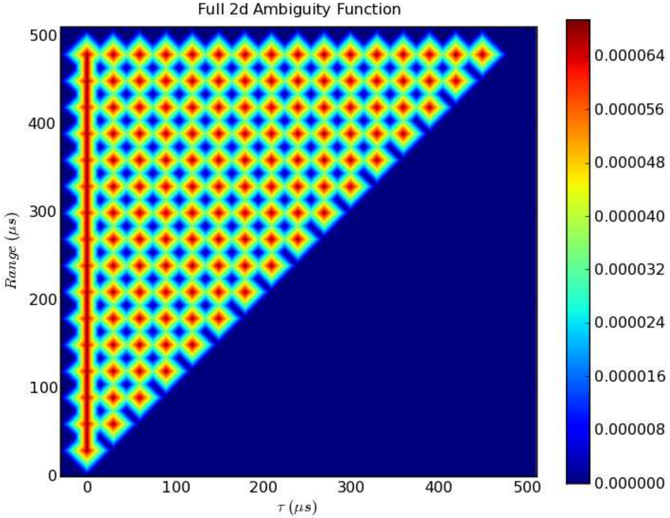


$$
\begin{aligned}
a_0 a_1 v_0 v_1^* =& a_0 \left(a_0 s_h^t + a_1 s_{h-1}^{t+\frac{1}{2}} + a_2 s_{h-2}^{t+1} + a_3 s_{h-3}^{t+\frac{3}{2}} \right) \times \\
& a_1 \left(a_0 s_{h+1}^{t+\frac{1}{2}} + a_1 s_h^{t+1} + a_2 s_{h-1}^{t+\frac{3}{2}} + a_3 s_{h-2}^{t+2} \right)^*
\end{aligned}
$$

$$
\begin{aligned}
E\left\{ a_0 a_1 v_0 v_1^* \right\} =& E\left\{ s_h^t s_h^{*t+1} \right\} + a_0 a_2 E\left\{ s_{h-1}^{t+\frac{1}{2}} s_{h-1}^{*t+\frac{3}{2}} \right\} \\
& + a_0 a_1 a_2 a_3 E\left\{ s_{h-2}^{t+1} s_{h-2}^{*t+2} \right\}
\end{aligned}
$$

Range-lag Ambiguity Function of Alternating Codes

Ambiguity function for a boxcar filter. 480 μs (16-baud, 30 μs baud, 32 pulse).



Canonical ISR Experiments

- D-region: Underspread pulse-to-pulse processing
- Perpendicular to **B** drifts (Jicamarca): Underspread pulse-to-pulse processing
- E-region: Alternating codes, overspread processing
- F-region and topside: Uncoded long pulse, overspread processing

Active area of research: Combination modes that compute lags both within the pulse (overspread) and pulse-to-pulse (underspread) in order to estimate the properties of the D-, E-, and F-regions simultaneously.