ISR Data Analysis and Fitting

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1. ISR Data Processing
   - Power Estimation
   - Underspread ACF Estimation
   - Overspread ACF Estimation
2. ISR Fitting
   - Parameter Estimation
   - Limitations on ISR Fitting
3. Derived ISR Data Products
   - Vector Winds and Electric Fields
   - Other Derived Parameters

ISR Data Processing

1. Collect baseband voltage samples
2. Form Power and ACF estimates (lag-products)
    - Requires averaging over many pulses
3. Fit ACFs for plasma parameters at each altitude using ISR theory
    - $N_e, T_e, T_i, V_{\mathrm{LOS}}$
    - Composition? Collision frequencies?
4. Process plasma parameters into higher level derived data products
    - Vector electric fields
    - Conductivities
    - Joule heating
    - Particle precipitation characteristics

$\vdots$

Power Estimation

Given K samples $v_i = s_i + n_i$, and an independently known noise power, N

$$\hat{S} = \frac{1}{K}\sum_{i=0}^{K-1} v_i v_i^* - N$$

$$E\left\{\hat{S}\right\} = S \qquad \text{unbiased estimator}$$

$$Var\left\{\hat{S}\right\} = \frac{1}{K}(S+N)^2$$

$$\frac{\delta\hat{S}}{S} = \frac{1}{\sqrt{K}}\left(1 + \frac{1}{S/N}\right)$$

For example, $\frac{\delta\hat{S}}{S} = 0.5$ with a $S/N = 0.1$ requires $K = 484$.
This assumes the samples are taken far apart and are uncorrelated.

Electron Density Determination

- ISR Power received (Watts)

$$P_{\mathrm{Rx}} = P_{\mathrm{Tx}} \frac{\tau_p}{2R^2} K_{sys} N_e \frac{2}{\left(1 + k^2 \lambda_{De}^2\right)\left(1 + k^2 \lambda_{De}^2 + T_e/T_i\right)}$$

$$\lambda_{De} = \sqrt{\frac{\epsilon_0 k_B T_e}{e^2 N_e}} \quad k = \frac{4\pi}{\lambda_{\mathrm{Tx}}} \quad R = \text{Range} \quad \tau_p = \text{Pulse Length (s)}$$

- P_{Rx} in Watts determined by comparing relative power received to direct signal injection (cal pulses)
- "System Constant" K_{sys} involves antenna gain, effective area, etc. For PFISR $K_{sys} \sim 10^{-19}\ \mathrm{m}^5\mathrm{s}^{-1}$.
- K_{sys} determined by comparing to absolute N_e measurements
    - Ionosonde f_{0F2}
    - ISR plasma line frequency
    - Faraday rotation (e.g. Jicamarca)

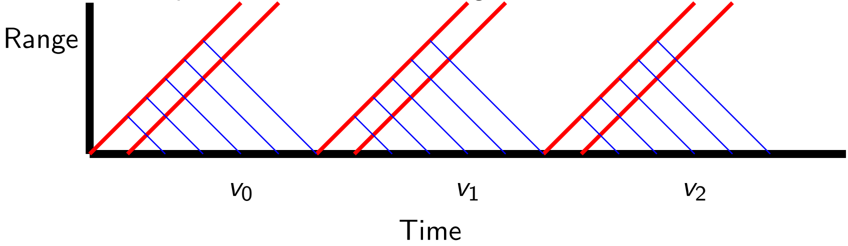
Reporting Electron Density

$$\text{Temperature Correction: } \zeta = \frac{2}{\left(1 + k^2\lambda_{De}^2\right)\left(1 + k^2\lambda_{De}^2 + T_e/T_i\right)}$$

- Uncorrected N_e: Assume $\zeta = 1$.
  - $T_e/T_i = 1$
  - $k^2\lambda_{De}^2 \ll 1$.
- N_e with model: Compute ζ using an empirical model of T_e/T_i as a function of altitude.
- N_e with fits: Compute ζ with T_e and T_i estimated from fitted ACF.

ACF Estimation (Pulse-to-Pulse)

Now assume pulses are taken close together and are correlated.



Unbiased Estimator:

$$\hat{R}_\ell = \frac{1}{K-\ell} \sum_{n=\ell}^{K-1} v_n v_{n-\ell}^*$$

$$E\left\{\hat{R}_\ell\right\} = R_\ell$$

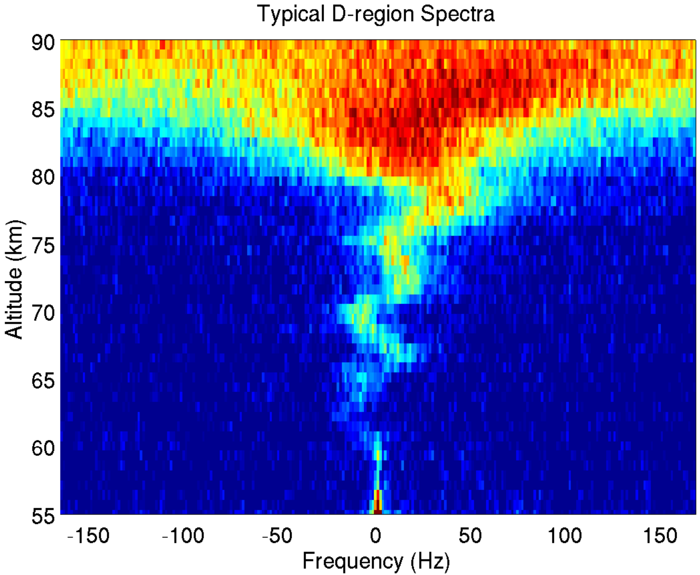
Biased Estimator:

$$\tilde{R}_\ell = \frac{1}{K} \sum_{n=\ell}^{K-1} v_n v_{n-\ell}^*$$

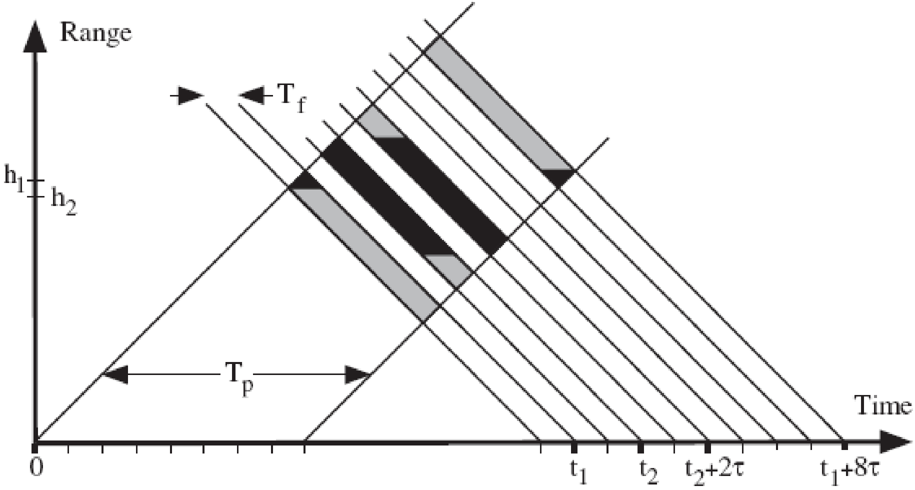
$$E\left\{\tilde{R}_\ell\right\} = \frac{K-\ell}{K} R_\ell$$

[triangular window]

Example D-region Spectra from PFISR

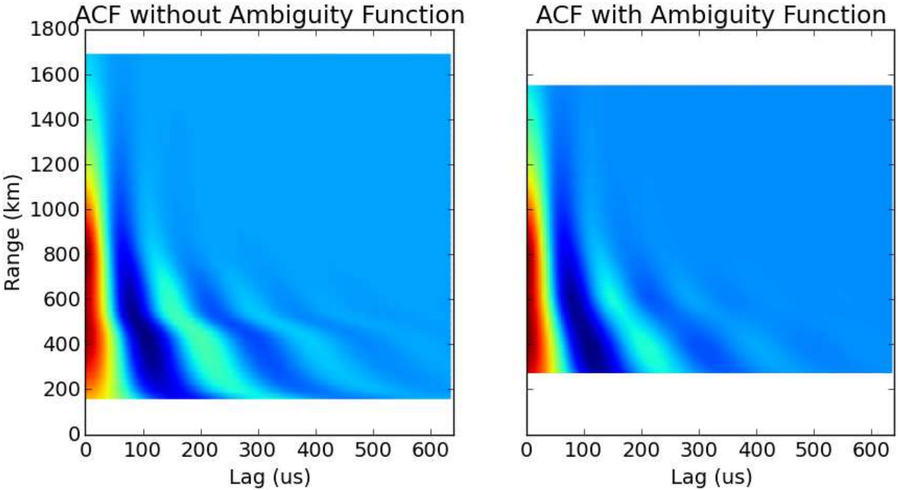


Uncoded Long Pulse

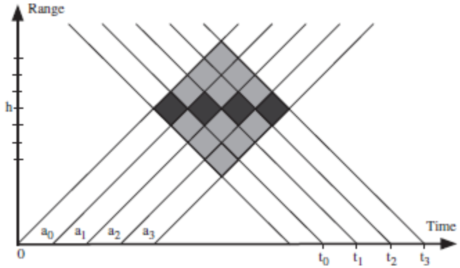


Blurring of ACFs by Ambiguity Functions

A particular exaggerated example using 1.5 ms long pulses and a profile with a sharp T_e gradient at 500 km.



Random Codes and Alternating Codes



$$a_0 a_1 v_0 v_1^* = a_0\left(a_0 s_h^t + a_1 s_{h-1}^{t+\frac{1}{2}} + a_2 s_{h-2}^{t+1} + a_3 s_{h-3}^{t+\frac{3}{2}}\right)\times$$
$$a_1\left(a_0 s_{h+1}^{t+\frac{1}{2}} + a_1 s_h^{t+1} + a_2 s_{h-1}^{t+\frac{3}{2}} + a_3 s_{h-2}^{t+2}\right)^*$$

$$E\left\{a_0 a_1 v_0 v_1^*\right\} = E\left\{s_h^t s_h^{*t+1}\right\} + a_0 a_2 E\left\{s_{h-1}^{t+\frac{1}{2}} s_{h-1}^{*t+\frac{3}{2}}\right\}$$
$$+ a_0 a_1 a_2 a_3 E\left\{s_{h-2}^{t+1} s_{h-2}^{*t+2}\right\}$$

Parameter Estimation and Inverse Problems

Given:

- Noisy measurements

$$\mathbf{Z} = \mathbf{Y} + \mathbf{W}$$

- The statistics of the noise

$$Cov\left\{\mathbf{W}\right\} = C$$

- A forward model for what the noiseless data should be for a given set of parameters β

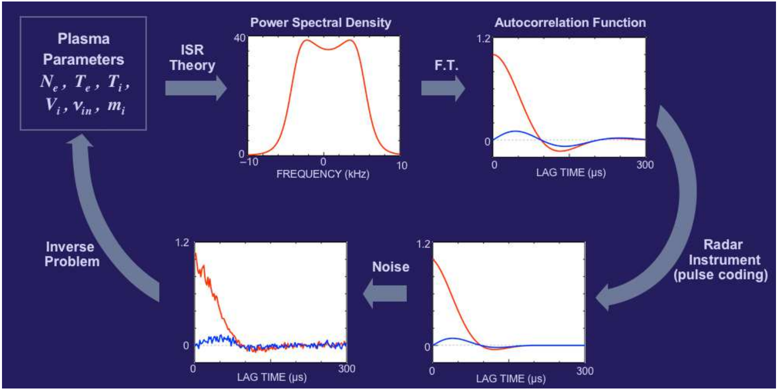
$$\mathbf{Y} = h\left(\beta\right)$$

How do I determine the best estimate of the parameters β?

Parameter Estimation Applied to ISR

For an ISR experiment:

- Noisy data are integrated lag-products (ACF estimators).
- Error properties determined from error analysis of estimators. Depends on SNR, self-clutter, etc.
- Parameters to be estimated are plasma state variables at each range:



Creating Forward Models

The forward model has two portions

1. Physics and Chemistry (ISR Theory)
    - Assume Maxwellian distributions?
    - Constraints on T_e and T_i?
    - Constraints on ion composition? Chemistry model?
    - Magnetic field effects
2. Instrumental Effects and Signal Processing
    - Sampling and Aliasing
    - Windowing
    - Ambiguity Functions

Best practice is the build the instrumental effects into the forward model. Do not manipulate the data in an attempt to undo the instrumental effects!

Least Squares Estimation

Least Squares Estimate:

$$\hat{\beta}_{\mathrm{LS}} = \min_{\beta} \left[h(\beta) - \mathbf{Z}\right]^T C^{-1} \left[h(\beta) - \mathbf{Z}\right]$$

For diagonal C

$$\hat{\beta}_{\mathrm{LS}} = \min_{\beta} \sum_i \frac{\left[h_i(\beta) - Z_i\right]^2}{\sigma_i^2}$$

- If $\mathbf{Z}$ is jointly gaussian, then the least-squares estimate is equivalent to the maximum likelihood estimate.
- A commonly used numerical technique for iteratively solving nonlinear least squares problems is the Levenberg-Marquardt algorithm
- Standard Levenberg-Marquart packages:
  - FORTRAN: MINPACK lmdif.f and lmder.f
  - Python: scipy.optimize.leastsq (wrapper around lmdif and lmder)
  - Matlab: Optimization Toolbox lsqnonlin
  - IDL: LMFIT
- Levenberg-Marquart requires a good initial guess

Error Propagation (Linear Least Squares)

Linear Least Squares $h(\beta) = H\beta$

$$\hat{\beta}_{\mathrm{LS}} = \left[H^T C^{-1} H\right]^{-1} H^T C^{-1}\mathbf{Z}$$
$$= \left[\tilde{H}^T \tilde{H}\right]^{-1} \tilde{H}^T \tilde{\mathbf{Z}}$$

where $\tilde{H} = C^{-1/2} H$ and $\tilde{\mathbf{Z}} = C^{-1/2}\mathbf{Z}$

Recall the property of jointly Gaussian random variables:

$$\mathbf{Y} = A\mathbf{X} \Rightarrow Cov\left\{\mathbf{Y}\right\} = ACov\left\{\mathbf{X}\right\} A^T$$

Thus

$$Cov\left\{\hat{\beta}_{\mathrm{LS}}\right\} = \left[\tilde{H}^T \tilde{H}\right]^{-1} \tilde{H}^T Cov\left\{\tilde{\mathbf{Z}}\right\} \tilde{H} \left[\tilde{H}^T \tilde{H}\right]^{-1}$$
$$= \left[\tilde{H}^T \tilde{H}\right]^{-1}$$

(Note $Cov\left\{\tilde{\mathbf{Z}}\right\} = C^{-1/2} Cov\left\{\tilde{\mathbf{Z}}\right\} C^{-1/2} = I$)

Error Propagation (Nonlinear Least Squares)

Suppose we are minimizing

$$\hat{\beta}_{\mathrm{LS}} = \min_{\beta} \sum_i \frac{\left[h_i(\beta) - Z_i\right]^2}{\sigma_i^2}$$

Linearize the problem in the vicinity of the final solution

$$Cov\left\{\hat{\beta}_{\mathrm{LS}}\right\} \approx \left[\tilde{J}^T \tilde{J}\right]^{-1}$$

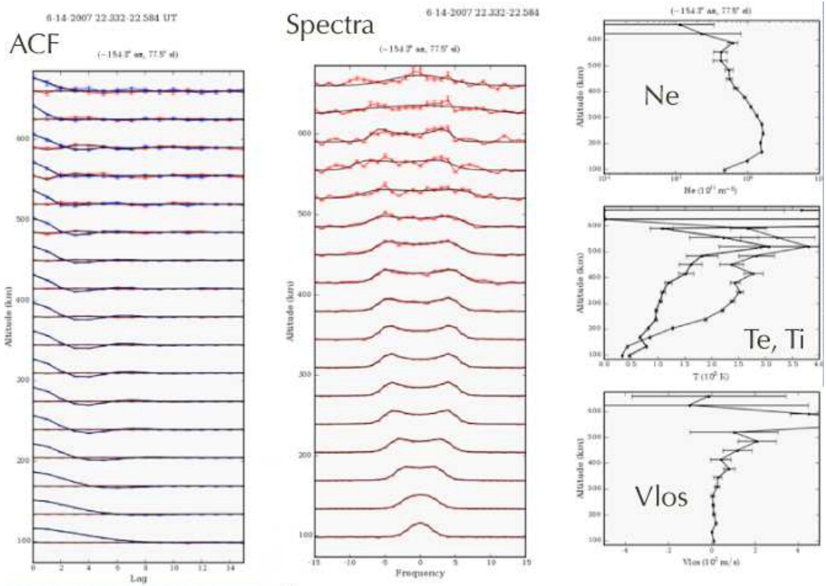
where the Jacobian $\tilde{J}$ is evaluated at the final solution $\beta = \hat{\beta}_{\mathrm{LS}}$

$$\tilde{J} = \begin{pmatrix} \frac{1}{\sigma_0}\frac{\partial h_0}{\partial \beta_0} & \frac{1}{\sigma_0}\frac{\partial h_0}{\partial \beta_1} & \cdots & \frac{1}{\sigma_0}\frac{\partial h_0}{\partial \beta_{M-1}} \\ \frac{1}{\sigma_1}\frac{\partial h_1}{\partial \beta_0} & \frac{1}{\sigma_1}\frac{\partial h_1}{\partial \beta_1} & \cdots & \frac{1}{\sigma_0}\frac{\partial h_1}{\partial \beta_{M-1}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\sigma_{N-1}}\frac{\partial h_{N-1}}{\partial \beta_0} & \frac{1}{\sigma_{N-1}}\frac{\partial h_{N-1}}{\partial \beta_1} & \cdots & \frac{1}{\sigma_{N-1}}\frac{\partial h_{N-1}}{\partial \beta_{M-1}} \end{pmatrix}$$

$\tilde{J}$ is $N \times M$ (tall and skinny)

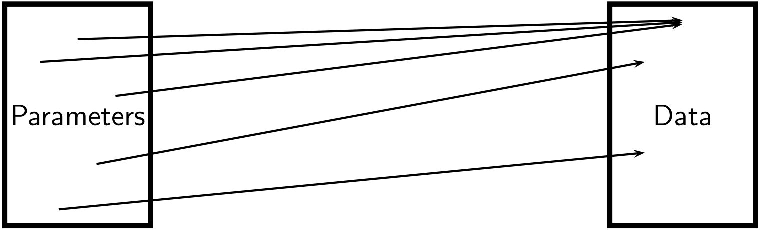
- Levenberg-Marquart computes $\tilde{J}$ at every iteration internally
- Standard packages usually have an option to return either $\tilde{J}$, and/or $\left[\tilde{J}^T \tilde{J}\right]^{-1}$ from the final iteration

Example PFISR Long Pulse Fits



Ill-Posed and Ill-Conditioned Problems

What happens if my forward model maps different points in parameter space to almost the same points in data space?



- **Ill-Posed Problem:** Multiple points in parameters space map to exactly the same point in data space.

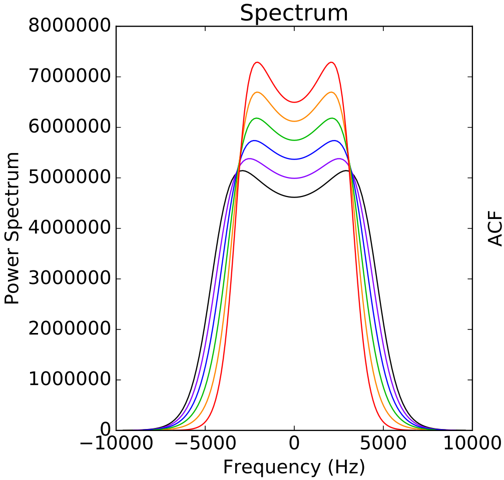
$$\left[\tilde{H}^T \tilde{H}\right] \text{ is singular, inverse problem is impossible}$$

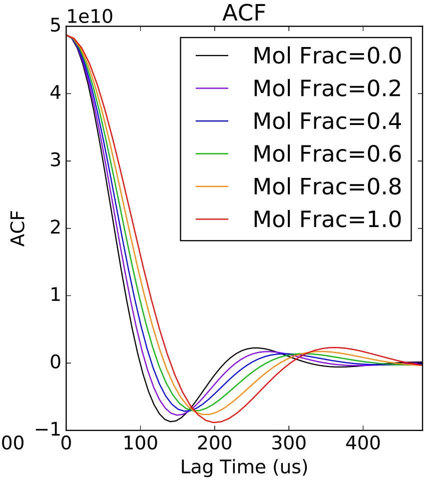
- **Ill-Conditioned Problem:** Multiple points in parameters space map to nearly the same point in data space.

$$\left[\tilde{H}^T \tilde{H}\right] \text{ is nearly singular, inverse problem is unstable given noisy data}$$

Ill-Conditioned ISR Theory: Molecular Ion Chemistry

Mixtures of O^+ and O_2^+ using $N_e = 10^{11}$, $T_e = T_i = 1000\ \mathrm{K}$

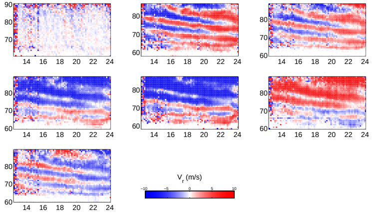




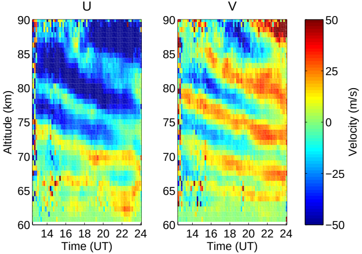
ISR spectrum measures $\sqrt{\frac{T_i}{m_i}}$, ambiguity between T_i and m_i

Mesospheric Vector Neutrals Winds

Line of Sight Velocities



Fitted Horizontal Velocities



$$\begin{pmatrix} V_{r,1} \\ \vdots \\ V_{r,7} \end{pmatrix} = \begin{pmatrix} \cos(\theta_1)\sin(\phi_1) & \cos(\theta_1)\sin(\phi_1) & \sin(\theta_1) \\ \vdots & \vdots & \vdots \\ \cos(\theta_7)\sin(\phi_7) & \cos(\theta_7)\sin(\phi_7) & \sin(\theta_7) \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

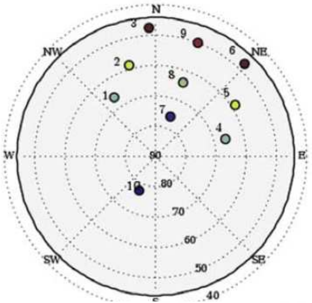
$$\mathbf{V}_r = \mathbf{DU}$$

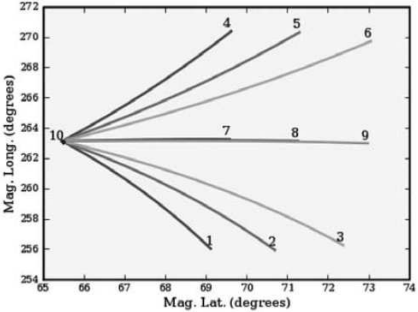
$$\mathbf{U} = (\mathbf{D}^{\mathrm{T}} C_{V_r}^{-1} \mathbf{D})^{-1} \mathbf{D}^{\mathrm{T}} C_{V_r}^{-1} \mathbf{V_r}$$

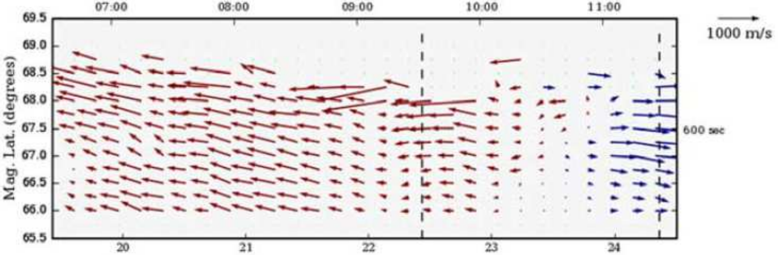
F-region 1-D Vector Electric Fields

- In F-region assume $\mathbf{v}_i = \frac{\mathbf{E}\times\mathbf{B}}{B^2}$
- Assume $\mathbf{E} \cdot \mathbf{B} = 0$ (no parallel fields)
- LOS velocity is related to **E** perpendicular to LOS and **B**
- Assume **E** is uniform in magnetic longitude, but varies with magnetic latitude
- Assume **E** fields map along equipotential field lines
- Different range gates correspond to different magnetic latitudes
- Fit for 2-components of **E** as a function of magnetic latitude

PFISR Electric Field Estimation







Interpretation of E-region Ion Velocities

Ion Momentum Equation:

$$0 = e\left(\mathbf{E} + \mathbf{u}_i \times \mathbf{B}\right) - m_i \nu_{in}\left(\mathbf{u}_i - \mathbf{u}_n\right)$$

Collisional Limit (D-region): $\mathbf{u}_i = \mathbf{u}_n$

Collisionless Limit (F-region): $\mathbf{u}_i = \dfrac{\mathbf{E} \times \mathbf{B}}{B^2}$

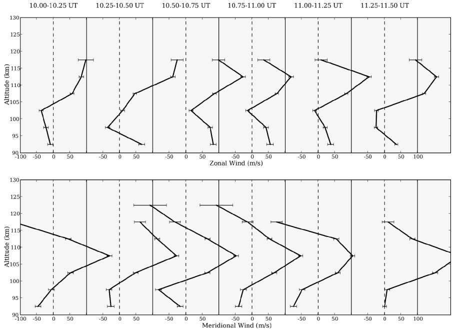
E-region: $\mathbf{u}_i = \begin{pmatrix} \frac{1}{1+\kappa_i^2} & \frac{-\kappa_i}{1+\kappa_i^2} & 0 \\ \frac{\kappa_i}{1+\kappa_i^2} & \frac{1}{1+\kappa_i^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \left[\mathbf{u}_n + \dfrac{e}{m_i \nu_{in}} \mathbf{E}\right]$

$$\kappa_i \equiv \frac{eB}{m_i \nu_{in}}$$

E-region Neutral Wind Estimation

- Estimate vector E-region ion velocities from E-region LOS velocity
- Estimate vector F-region electric fields from F-region LOS velocity
- Map electric fields from F-region to E-region along equipotential field lines
- Solve for $\mathbf{u}_n$

$$\mathbf{u}_n = \mathbf{u}_i - \frac{e}{m_i \nu_{in}} (\mathbf{E} + \mathbf{u}_i \times \mathbf{B})$$



Heinselman and Nicolls (2008) Radio Sci.

Derived Electrodynamical Parameters

- Conductivity

$$\sigma_P = N_e e^2 \left(\frac{\nu_{en}/m_e}{\nu_{en}^2 + \Omega_e^2} + \frac{\nu_{in}/m_i}{\nu_{in}^2 + \Omega_i^2} \right)$$
$$\sigma_H = N_e e^2 \left(\frac{\Omega_e/m_e}{\nu_{en}^2 + \Omega_e^2} - \frac{\Omega_i/m_i}{\nu_{in}^2 + \Omega_i^2} \right)$$

- Horizontal Currents

$$\mathbf{J} = \sigma_P \left(\mathbf{E} + \mathbf{u}_n \times \mathbf{B}\right) - \sigma_H \left[\left(\mathbf{E} + \mathbf{u}_n \times \mathbf{B}\right) \times \frac{\mathbf{B}}{B}\right]$$

- Joule Heating

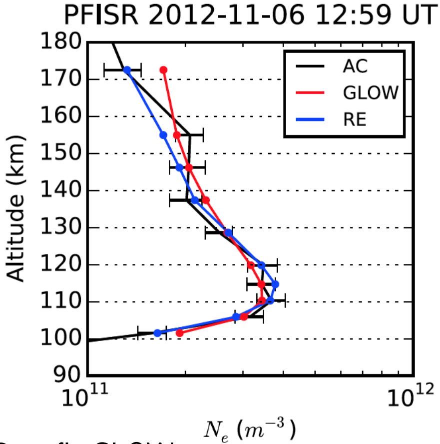
$$Q_J = \mathbf{J} \cdot \left(\mathbf{E} + \mathbf{u}_n \times \mathbf{B}\right)$$
$$= \sigma_P \left|\mathbf{E} + \mathbf{u}_n \times \mathbf{B}\right|^2$$

See Thayer (1998) *JGR* and Thayer and Semeter (2004) *JASTP*

Precipitation Characteristics from N_e Profile Inversion

- Input N_e profiles vs altitude (up-B beam)
- Estimate precipitating energy flux and characteristic energy
- Use a forward model of energetic electron transport, impact ionization, and recombination (e.g. GLOW).

Kaeppler et al. (2015) *JGR*.



Best fit GLOW parameters:
$Q_0 = 7.3 \pm 0.8 \ \mathrm{mW/m^2}$,
$E_0 = 5.0 \pm 0.2 \ \mathrm{keV}$

Ongoing Research Areas

Researchers are continuing to innovate and find new ways to extract more information from ISR data.

- Ion outflow fluxes
- Topside and plasmasphere parameters
- Neutral temperature derived from T_e and T_i
- Neutral density derived from multi-frequency ISR
- Electron and ion heat fluxes
- Ion temperature anisotropy
- Non-Maxwellian distribution functions
- Gravity wave frequencies and wavevectors
- Energetic electron distributions from plasma line powers

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