Rocket Kalman filter formulas

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1 Coordinates

x, y and z are coordinates in the reference frame of the ground.
x', y' and z' are coordinates in the reference frame of the rocket.

2 Time step for coordinates and velocities

In time step $t \to t + \Delta t$, we estimate the next state of the system as

$$\vec{r}_1 = \vec{r} + \vec{v}\Delta t + \vec{a}\frac{(\Delta t)^2}{2}$$

$$\vec{v}_1 = \vec{v} + \vec{a}\Delta t$$

We already have $\vec{r}$ and $\vec{v}$, but we have to estimate $\vec{a}$ from $\vec{r}$, $\vec{v}$ and t. Therefore, we could write

$$(\vec{r}_1, \vec{v}_1) = f(\vec{r}, \vec{v}, t)$$

We also have to compute the Jacobian (F) of f with respect to all components of $\vec{r}$ and $\vec{v}$ over all their components (a 6×6 matrix).

3 Acceleration

$$\vec{F} = \vec{F}_{drag} + \vec{F}_{gravity} + \vec{F}_{thrust}$$

$$\vec{F}_{drag} = -\frac{1}{2}c_{drag}(v)\rho(z)v^2\hat{z}'$$

Despite drag is in the $\hat{v}$ direction, it is safe to assume that $\hat{v} = \hat{z}$ because of a great restoring force.

$$\vec{F}_{gravity} = -mg\hat{z}$$

Rearranging for acceleration gives

$$\vec{a} = \frac{\vec{F}_{thrust} - \frac{1}{2}c_{drag}(v)\rho(z)v^2}{m(t)} \cdot \hat{z}' - g \cdot \hat{z}$$

where

$$c_{drag} = c_{body} \cdot A$$

c_{drag} comes from RASAero Cd(M) graph - we wil have a lookup table for $0 \leq M < 5$ with step 0.01.

4 Acceleration with respect to coordinates

The only part of acceleration that depends on coordinates of the rocket is the density, which only depends on the height of the rocket, therefore

$$\frac{\delta \vec{a}}{\delta x} = \frac{\delta \vec{a}}{\delta y} = 0$$

$$\frac{\delta \vec{a}}{\delta z} = -\frac{c_{drag}(v)v^2}{2m(t)} \frac{\delta \rho}{\delta z} \cdot \hat{z}'$$

Given

$$\rho(h) = \begin{cases} 1.1621 - 0.00007 * h, & \text{if } h \leq 13000m \\ 0.25, & \text{otherwise} \end{cases}$$

it immediately follows that

$$\frac{\delta \rho}{\delta z} = \begin{cases} -0.00007, & \text{if } z \leq 13000m \\ 0, & \text{otherwise} \end{cases}$$

Should be improved, as there is a gap at $h = 13000m$

Linearized Decay of Density w.r.t. Altitude - Source

5 Acceleration with respect to velocities

We will mark coordinate derivatives $\dot{q}$. // explain q dot

$$\frac{\delta \vec{a}}{\delta \dot{q}} = -\frac{\rho}{2m(t)} \frac{\delta (c_{drag}(v)v^2)}{\delta \dot{q}} \cdot \hat{z}'$$

$$\frac{\delta \vec{a}}{\delta \dot{q}} = -\frac{\rho}{2m(t)} (\frac{\delta c_{drag}}{\delta v} v^2 + 2c_{drag}v) \cdot \frac{\delta v}{\delta \dot{q}} \cdot \hat{z}'$$

Just to recall, $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$,, so

$$\frac{\delta v}{\delta \dot{q}} = \frac{\dot{q}}{v}$$

Thus

$$\frac{\delta \vec{a}}{\delta \dot{q}} = -\frac{\rho}{2m(t)}(\frac{\delta c_{drag}}{\delta v}v^2 + 2c_{drag}v) \cdot \frac{\dot{q}}{v} \cdot \hat{z}' = -\frac{\rho}{m(t)}(\frac{\delta c_{drag}}{\delta v}\frac{v}{2} + c_{drag}) \cdot \dot{q}\hat{z}'$$

6 F solution

$$\begin{pmatrix} \frac{dt^2}{2}\frac{\delta a}{\delta z}R_{z'q_i}\delta_i^z & Idt + \frac{dt^2}{2}\frac{\delta a}{\delta \dot{q}_i}R_{z'q_i} \\ dt\frac{\delta a}{\delta z}R_{z'q_i}\delta_i^z & dt\frac{\delta a}{\delta \dot{q}_i}R_{z'q_i} \end{pmatrix}$$

$$\frac{\delta \vec{a}}{\delta \dot{q}} = -\frac{\rho}{m(t)}(\frac{\delta c_{drag}}{\delta v}\frac{v}{2} + c_{drag}) \cdot \dot{q}\hat{z}'$$

$$\frac{\delta \vec{a}}{\delta z} = -\frac{c_{drag}(v)v^2}{2m(t)}\frac{\delta \rho}{\delta z} \cdot \hat{z}'$$

7 Full Jacobian F

From Section 2 we can get

$$\frac{\delta \vec{r}_1}{\delta q} = \frac{\delta \vec{r}}{\delta q} + \Delta t\frac{\delta \vec{v}}{\delta q} + \frac{(\Delta t)^2}{2}\frac{\delta \vec{a}}{\delta q}$$

$$\frac{\delta \vec{v}_1}{\delta q} = \frac{\delta \vec{v}}{\delta q} + \Delta t\frac{\delta \vec{a}}{\delta q}$$

Knowing that $\frac{\delta \vec{v}}{\delta q} = 0$ (velocity doesn't depend on the coordinate) and $\frac{\delta \vec{r}}{\delta q} = \hat{q}$, we get

$$\frac{\delta \vec{r}_1}{\delta q} = \hat{q} + \frac{(\Delta t)^2}{2}\frac{\delta \vec{a}}{\delta q}$$

$$\frac{\delta \vec{v}_1}{\delta q} = \Delta t\frac{\delta \vec{a}}{\delta q}$$

Similarly, for $\dot{q}$ derivatives:

$$\frac{\delta \vec{r}_1}{\delta \dot{q}} = \frac{\delta \vec{r}}{\delta \dot{q}} + \Delta t\frac{\delta \vec{v}}{\delta \dot{q}} + \frac{(\Delta t)^2}{2}\frac{\delta \vec{a}}{\delta \dot{q}}$$

$$\frac{\delta \vec{v}_1}{\delta \dot{q}} = \frac{\delta \vec{v}}{\delta \dot{q}} + \Delta t\frac{\delta \vec{a}}{\delta \dot{q}}$$

With similar reasoning, $\frac{\delta \vec{r}}{\delta \dot{q}} = 0$ and $\frac{\delta \vec{v}}{\delta \dot{q}} = \hat{\dot{q}}$

$$\frac{\delta \vec{r}_1}{\delta \dot{q}} = \Delta t \hat{\dot{q}} + \frac{(\Delta t)^2}{2} \frac{\delta \vec{a}}{\delta \dot{q}}$$

$$\frac{\delta \vec{v}_1}{\delta \dot{q}} = \hat{\dot{q}} + \Delta t \frac{\delta \vec{a}}{\delta \dot{q}}$$

8 Prandtl-Meyer

Now for the velocity-dependent Prandtl-Meyer corrections. We have, for diatomic gasses, the factor as a function of the rocket's and the measured Mach numbers $M = v/v_{mach}$

$$f = \frac{P_{baro}}{P_{atm}} = (\frac{1 + 0.2M_{rocket}^2}{1 + 0.2M_{meas}^2})^{3.5}$$

Where $M_{meas} = \nu^{-1}(\nu(M) - \alpha)$
Valid when $\nu(M) > \alpha$ The barometer class uses the following model for height h (m) given a pressure reading P from a reference baseline P_0:

$$h = 44330 \left(1 - (P/P_0)^{1/5.255}\right)$$

$$P = P_0(1 - h/44330)^{5.255}$$

By rearrangement, we find the expected height measurement given a new effective pressure fP

$$h_{meas}(fP(h_{atm})) = 44330 \left(1 - (f(1 - h/44330)^{5.255})^{1/5.255}\right) = 44330(1-f^{0.19})+hf^{0.19}$$

For the barometer measurement with the Prandtl-Meyer function ν at angle α, we calculate h_{meas} for our h vector using

$$h_{meas} = 44330(1 - f(M)^{0.19}) + zf(M)^{0.19}$$

$$M = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}/v_{mach}$$

$$f = (\frac{1 + 0.2M^2}{1 + 0.2\nu^{-1}(\nu(M) - \alpha)^2})^{3.5}$$

$$\frac{\partial h_{meas}}{\partial z} = f^{0.19}$$

For the velocity dependence in the H matrix:

$$\frac{\partial h_{meas}}{\partial \dot{q}_i} = \frac{\partial h_{meas}}{\partial M}\frac{\partial M}{\partial \dot{q}_i} = \frac{\partial h_{meas}}{\partial M}\frac{\partial M}{\partial v}\frac{\partial v}{\partial \dot{q}_i} = \frac{1}{v_{mach}}\frac{\dot{q}_i}{v}\frac{\partial h_{meas}}{\partial f}\frac{\partial f}{\partial M}$$

$$\frac{\partial h_{meas}}{\partial f} = -(44330.0 - h) * 0.19 * f^{-0.81}$$

$$\frac{\partial f}{\partial M} = 3.5(\frac{1 + 0.2M^2}{1 + 0.2M_{meas}^2})^{2.5}\frac{0.4M(1 + 0.2M_{meas}^2) - (1 + 0.2M^2)0.4M_{meas}\frac{\partial M_{meas}(M)}{\partial M}}{(1 + 0.2M_{meas}^2)^2}$$

$$= 3.5f^{2.5/3.5}(\frac{0.4M}{1+0.2M_{meas}^2} - \frac{f0.4M_{meas}}{1+0.2M_{meas}^2}\frac{\partial M_{meas}(M)}{\partial M})$$

Now we're left at

$$\frac{\partial M_{meas}(M)}{\partial M} = \frac{\partial \nu^{-1}(\nu(M)-\alpha)}{\partial M}$$

Filled in from our simplest possible ν implementation/linear approximation $1 < M < 4.5$:

$$\nu(M) = \sqrt{6}\arctan\sqrt{\frac{M^2-1}{6}} - \arctan\sqrt{M^2-1}$$

$$\nu(M) \approx \begin{cases} 0.4(M-1) & M > 1 \\ 0 & M \leq 1 \end{cases}$$

$$\nu^{-1}(\nu) \approx \begin{cases} 2.5\nu + 1 & \nu > 0 \\ 1 & \nu \leq 0 \end{cases}$$

Then

$$\frac{\partial M_{meas}(M)}{\partial M} = \frac{\partial}{\partial M}(2.5(0.4M - 0.4 - \alpha) + 1) = 1$$

These are the current/bad implementation:

$$\nu(M) \approx \begin{cases} 7.6(M-0.9) & M < 1.16 \\ 1.92 + \sqrt{0.1M - 0.115} & M \geq 1.16 \end{cases}$$

$$\nu^{-1}(\nu) \approx \begin{cases} 0.1315789 * \nu + 0.9 & \nu < 1.94 \\ 1.15 + 10 * (\nu - 1.92)^2 & \nu \geq 1.94 \end{cases}$$

Ignoring that one,

$$f = \left(\frac{1+0.2M^2}{1+0.2(M-0.25\alpha)^2}\right)^{3.5}$$

$$\frac{\partial h_{meas}}{\partial \dot{q}_i} = \frac{0.266(z-44330.0)}{v_{mach}}\frac{\dot{q}_i}{v}f^{-0.096}(\frac{M - fM_{meas}}{1+0.2M_{meas}^2})$$

Which leaves us with the last 3 of our 4 needed Jacobian entries:

$$\frac{\partial h_{meas}}{\partial \dot{q}_i} = \frac{0.266(z-44330.0)}{v_{mach}}\frac{\dot{q}_i}{v}\left(\frac{1+0.2M^2}{1+0.2(M-0.25\alpha)^2}\right)^{-0.336}\frac{M - \left(\frac{1+0.2M^2}{1+0.2(M-0.25\alpha)^2}\right)^{3.5}(M-0.25\alpha)}{1+(M-0.25\alpha)^2}$$

The rest of the Jacobian entries are [from the end of sections 5 and 6]

$$\frac{\partial a_i}{\partial z} = -\frac{c_{drag}(v)v^2}{2m(t)}\frac{\delta\rho}{\delta z}\cdot R_{zq'_i}$$

$$\frac{\partial a_i}{\partial \dot{q}_j} = -\frac{\rho}{m(t)}(\frac{\delta c_{drag}}{\delta v}\frac{v}{2} + c_{drag})R_{q_j q'_i}$$

9 JUNK

Junk factor for

$$\frac{\partial h_{meas}}{\partial \dot{q}_i}$$

$$B = (probably 0) = \frac{\partial h_{meas}}{\partial z}\frac{\partial z}{\partial \dot{q}_i} = \left(\frac{1 + 0.2M^2}{1 + 0.2(M - 0.25\alpha)^2}\right)^{0.19*3.5}\left(dt - \frac{dt^2\rho}{2*2m(t)}(\frac{\delta c_{drag}}{\delta v}v + 2c_{drag}) \cdot \dot{q}_i\right)$$