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ON THE MASSES OF NEBULAE AND OF CLUSTERS OF NEBULAE

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ABSTRACT

Present estimates of the masses of nebulae are based on observations of the *luminosities* and *internal rotations* of nebulae. It is shown that both these methods are unreliable; that from the observed luminosities of extragalactic systems only lower limits for the values of their masses can be obtained (sec. i), and that from internal rotations alone no determination of the masses of nebulae is possible (sec. ii). The observed internal motions of nebulae can be understood on the basis of a simple mechanical model, some properties of which are discussed. The essential feature is a central core whose internal *viscosity* due to the gravitational interactions of its component masses is so high as to cause it to rotate like a solid body.

In sections iii, iv, and v three new methods for the determination of nebular masses are discussed, each of which makes use of a different fundamental principle of physics.

Method iii is based on the *virial theorem* of classical mechanics. The application of this theorem to the Coma cluster leads to a minimum value $\overline{M}=4.5\times 10^{10}M_{\odot}$ for the average mass of its member nebulae.

Method iv calls for the observation among nebulae of certain *gravitational lens* effects.

Section v gives a generalization of the principles of ordinary *statistical mechanics* to the whole system of nebulae, which suggests a new and powerful method which ultimately should enable us to determine the masses of all types of nebulae. This method is very flexible and is capable of many modes of application. It is proposed, in particular, to investigate the distribution of nebulae in individual great clusters.

As a first step toward the realization of the proposed program, the Coma cluster of nebulae was photographed with the new 18-inch Schmidt telescope on Mount Palomar. Counts of nebulae brighter than about $m=16.7$ given in section vi lead to the gratifying result that the distribution of nebulae in the Coma cluster is very similar to the distribution of luminosity in globular nebulae, which, according to Hubble's investigations, coincides closely with the theoretically determined distribution of matter in isothermal gravitational gas spheres. The high central condensation of the Coma cluster, the very gradual decrease of the number of nebulae per unit volume at great distances from its center, and the hitherto unexpected enormous extension of this cluster become here apparent for the first time. These results also suggest that the current classification of nebulae into relatively few *cluster nebulae* and a majority of *field nebulae* may be fundamentally inadequate. From the preliminary counts reported here it would rather follow that practically *all* nebulae must be thought of as being grouped in clusters—a result which is in accord with the theoretical considerations of section v.

In conclusion, a comparison of the relative merits of the three new methods for the determination of nebular masses is made. It is also pointed out that an extensive investigation of great clusters of nebulae will furnish us with decisive information regarding the question whether physical conditions in the known parts of the universe are merely fluctuating around a stationary state or whether they are continually and systematically changing.

The determination of the masses of extragalactic nebulae constitutes at present one of the major problems in astrophysics. Masses of nebulae until recently were estimated either from the luminosities of nebulae or from their internal rotations. In this paper it will be shown that both these methods of determining nebular masses are unreliable. In addition, three new possible methods will be outlined.

I. MASSES FROM LUMINOSITIES OF NEBULAE

The observed absolute luminosity of any stellar system is an indication of the approximate amount of luminous matter in such a system. In order to derive trustworthy values of the masses of nebulae from their absolute luminosities, however, detailed information on the following three points is necessary.

1. According to the mass-luminosity relation, the conversion factor from absolute luminosity to mass is different for different types of stars. The same holds true for any kind of luminous matter. In order to determine the conversion factor for a nebula as a whole, we must know, therefore, in what proportions all the possible luminous components are represented in this nebula.

2. We must know how much dark matter is incorporated in nebulae in the form of cool and cold stars, macroscopic and microscopic solid bodies, and gases.

3. Finally, we must know to what extent the apparent luminosity of a given nebula is diminished by the internal absorption of radiation because of the presence of dark matter.

Data are meager[1] on point 1. Accurate information on points 2

[1] It should, however, be mentioned that certain spiral nebulae seem to be stellar systems similar in composition to the local Kapteyn system of our galaxy. For such systems the conversion factors may with some confidence be set equal to the conversion factor of the Kapteyn system. See also E. Hubble, *Ap. J.*, **64**, 148, 1929.

and 3 is almost entirely lacking. Estimates of the masses of nebulae from their observed luminosities are therefore incomplete and can at best furnish only the lowest limits for the values of these masses.

II. MASSES FROM INTERNAL ROTATIONS OF NEBULAE

It has apparently been taken for granted by some astronomers[2] that from observations on the internal rotations good values for the masses M_N of nebulae could be derived. Values of the order of $M_N = 10^9\ M_\odot$ up to $M_N = 4 \times 10^{10}\ M_\odot$ were obtained in this way,[2] where $M_\odot = 2 \times 10^{33}$ gr is the mass of the sun. A closer scrutiny of the behavior of suitably chosen mechanical models of stellar systems, unfortunately, soon reveals the fact that the masses of such systems, for a given distribution of average angular velocities throughout the system, are highly indeterminate, and vice versa. This conclusion may, for instance, be derived from the consideration of two limiting models of a nebula as a mechanical system.

A. MODEL OF A NEBULA WHOSE "INTERNAL VISCOSITY" IS NEGLIGIBLE

This model consists of a heavy and small nucleus of mass M_0 around which a given number, n, of stars of average mass $M_s \ll M_0/n$ describe planetary orbits. The mutual gravitational interactions between these outlying stars are negligible, and the system may therefore be said to have an internal "viscosity" equal to zero. It is obvious that under these circumstances we may build up models that satisfy almost any specifications in regard to total mass, total luminosity, and internal distribution of luminosity as well as distribution of the average angular velocities. We may, for instance, distribute our n stars over the six-dimensional manifold of all possible planetary orbits (including the epochs or phases) in such fashion that the average angular velocity of the resulting system S_0 is zero in every point. Since all these orbits are essentially non-interacting, we may reverse the sense of rotation (direction of stellar motion) in an arbitrary number of these orbits. In this way a system S of specified distribution of average angular velocities may be constructed whose remaining characteristics, such as the mass, the luminosity,

[2] E. Hubble, *The Realm of Nebulae* (New Haven: Yale University Press, 1936), p. 179; also *Ap. J.*, **69**, 150, 1929.

and the external shape, are identical with those of S_0. Thus, the observed angular velocities in themselves give no clue regarding the mass of the system.

B. MODEL OF A NEBULA WHOSE INTERNAL VISCOSITY IS VERY GREAT

This model is built up of stars, dust, and gases in such fashion that the gravitational interactions between the various components, as well as direct impacts, influence the path of every component mass in a radical way. Many changes in energy and momentum of every component mass will take place during time intervals that are short compared with the time an unperturbed mass of the same initial velocity would consume to traverse the system.

Conditions of motion in this model are analogous to the conditions of motion of elementary particles in a star. This model of a nebula, therefore, will rotate like a solid body, regardless of what its total mass and the distribution of mass over different regions of the system may be. The conclusion which has sometimes been put forward,[2] that constant angular velocity necessarily implies uniform distribution of mass, is obviously erroneous. Furthermore, it is again seen that the rate of rotation of a stellar system has no very direct bearing on its total mass.

C. ACTUAL NEBULAE

Good mechanical models of actual nebulae may presumably be constructed by combining the distinctive features of the two limiting cases described in the preceding sections. Such a combined model will possess a central, highly viscous core whose relative dimensions are not negligible but are comparable with the extension of the whole system. If the outlying, and among themselves little interacting, components of the nebula had no connection with the central core, we might, at a given instant, observe average angular velocities Ω which, as a function of the distance r from the center of rotation, would be given by

$$\Omega(r) = \Omega_0 = \text{const.} \longleftrightarrow \text{for } r < r_0 \,, \tag{1}$$

where r_0 is the radius of the core. For $r > r_0$, the angular velocity $\Omega(r)$ would be essentially arbitrary. In reality, however, the viscosity will not drop abruptly to zero at $r = r_0$. From an inspection of the distribution of the outlying masses in many nebulae it would seem that these masses at some previous time must have formed part of the central core. They may have been ejected from this core because they acquired high kinetic energy through many close encounters, or they may be the result of a partial disruption of the core by tidal actions caused by encounters with other nebulae (Jeans). At the moment these outlying masses have passed the boundary of the core, their average tangential velocities must have been of the order $r_0\Omega_0$. Assuming that these masses in regions $r > r_0$ are essentially subject only to the gravitational attraction of a central spherical core and that the interactions among themselves may be neglected (the internal viscosity of the outlying system is equal to zero), the average angular velocity $\Omega(r)$ outside the core will be approximately given as

$$\Omega(r) = \frac{r_0^2\Omega_0}{r^2} \qquad \text{for } r > r_0 \, . \tag{2}$$

This relation simply expresses the fact that a mass m which, on being ejected from the core with a tangential velocity v_t relative to this core, describes an orbit whose angular momentum,

$$mr^2\omega = m[r_0 v_t + r_0^2\Omega_0] = c \, , \tag{3}$$

is a constant. If the average $\overline{v_t}$ for many particles leaving the core is zero, the average angular velocity $\overline{\omega} = \Omega$ of all the particles in a given point is obtained from

$$r^2\overline{\omega} = r^2\Omega(r) = r_0^2\Omega_0 \, , \tag{4}$$

which is the same as (2). Unfortunately, relation (2) is superficially very similar to the relation obeyed by the angular velocities ω_c of a system of circular planetary orbits around a heavy central mass M. For such an orbit we have

$$mr\omega_c^2 = \frac{\Gamma mM}{r^2} \tag{5}$$

or

$$\omega_c = \frac{(\Gamma M)^{1/2}}{r^{3/2}} \, , \tag{6}$$

where Γ is the universal gravitational constant. The similarity of the dependence on r in (2) and (6) will in reality become still greater, since the rate of decrease of $\Omega(r)$ with increasing values of r will in actual nebulae be more gradual than that given by (2), owing to the fact that the internal viscosity will not vanish abruptly at $r = r_0$ but will disappear gradually with increasing r. The observed angular velocities in the outlying regions of nebulae actually show a dependence on r which resembles the relation (6). This relation was, therefore, sometimes erroneously used for the determination of the mass M. The preceding discussion, however, indicates that the observed angular velocities may be adequately accounted for on the basis of the considerations resulting in relation (2) rather than in relation (6). This again shows clearly that it is not possible to derive the masses of nebulae from observed rotations without the use of additional information, since the relation (2) does not contain the mass M at all.

D. FURTHER DISCUSSION OF THE ROTATION OF NEBULAE

From the analysis of an appropriate mechanical model of actual nebulae we have derived in the preceding section the following approximate dependence on r of the average tangential velocity V_t:

$$V_t = r\Omega_0 \qquad \text{for } r < r_0 \tag{7}$$

$$V_t = \frac{r_0^2\Omega_0}{r} \qquad \text{for } r > r_0\,. \tag{8}$$

This dependence is pictured in Figure 1 . The observed curves $V_t(r)$ are similar to the schematic curve of Figure 1. From this similarity we infer that the cores of nebulae possess considerable internal viscosity. The problem of deducing theoretical values for this viscosity therefore arises. It will be interesting to see whether the gravitational interactions in dense stellar systems are sufficient to account for the fact that such systems rotate like solid bodies, or if it is necessary to introduce matter in the form of dust particles and gases.

The point of view adopted in the preceding discussion automatically eliminates the discrepancy which was thought to exist, with respect to the distribution of mass in nebulae as derived from data on internal rotations, on the one side, and from the shape and the distribution of the isophotal contours, on the other. The tremendous increase of surface brightness from the edge, $r = r_0$, of the core of nebulae to their center, $r = 0$, indicates a correspondingly large increase of mass density. The erroneous idea[2] that the constancy of the angular velocity throughout the core necessitates the assumption of a constant mass density therefore created an apparently insoluble paradox. This paradox, however, disappears as soon as we introduce the idea of an internal gravitational viscosity of stellar systems, which equalizes the angular velocity throughout such systems regardless of the distribution of mass. How this viscosity may be expressed in terms of the gravitational interactions of stars will be discussed in another place.

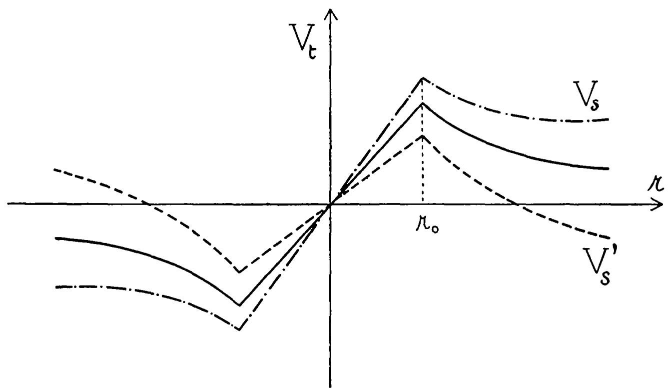

Fig. 1.—Velocity of rotation in nebulae

One further interesting problem presents itself. The distribution of matter in a stellar system which has an internal viscosity as high as we have assumed it to be in the cores of nebulae should rapidly converge toward stationary conditions. The determination of the density distribution in such a system should be analogous to the determination of the density distribution in gravitating gas spheres. R. Emden's analysis[3] of such spheres may, therefore, prove useful in the study of globular and elliptical nebulae as well as in the study of cores of spiral nebulae.

In concluding this section the question may be raised whether

[3] *Gaskugeln* (Leipzig, 1907).

data on the internal rotation of nebulae, supplemented by certain additional information, make possible the determination of nebular masses. This question, in principle, must be answered in the affirmative. For instance, data on internal velocities combined with the virial theorem discussed in the next section may ultimately furnish good values for the masses of globular and elliptical nebulae as well as the cores of certain spirals. In the case of open spirals, an investigation of the geometrical structure of the spiral arms, combined with

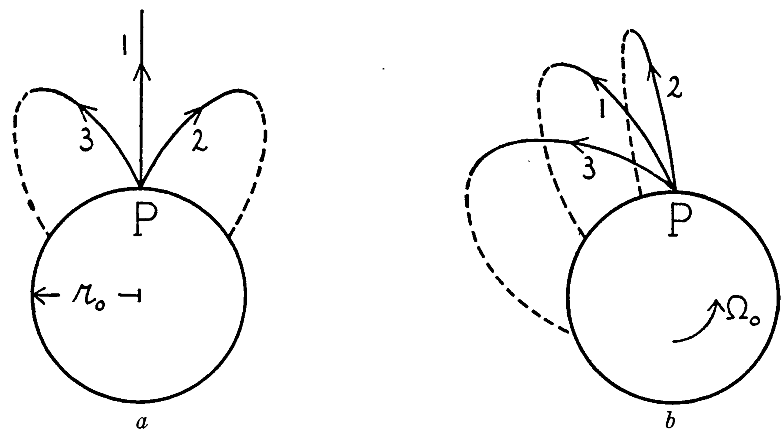

FIG. 2.—Rotation of nebulae and shape of spiral arms

velocity data, promises to be helpful. In this connection a few remarks may be in order regarding the possibility of the formation of spiral arms in our simple mechanical model of nebulae.

Consider the two alternatives pictured in Figure 2: (*a*) a nonrotating core, that is, $\Omega_0 = 0$, and (*b*) a rotating core, $\Omega_0 \neq 0$. Suppose that some disturbance, such as tidal action, causes the ejection from the core of a number of stars at *P*. The resulting orbits of three stars, 1, 2, and 3, are schematically indicated in Figure 2. For $\Omega_0 \neq 0$ distinct spiral arms may be formed, provided a sufficient number of stars are ejected whose radial and tangential velocity components v_r and v_t relative to the core are in certain favorable relations to the peripheral velocity $r_0 \Omega_0$ of the core.

It should be remarked that the orbits 1, 2, 3, etc., in Figure 2*b* are ellipses only if outside the core the corresponding masses are no longer acted upon by the forces which originally caused their ejection from the core. If, for example, the ejection is brought about by the close encounter of the nebula with another nebula, tidal actions will continue until the second nebula has moved far away and our simple model must be correspondingly modified. It may suffice here to point out that in addition to the creation of angular momentum in a nebula, a close encounter may also set up radial pulsation in this nebula. This pulsation will not, in general, be spherically symmetrical. As a first approximation we can schematically characterize it by what might be called the "tidal ellipsoid" or "ellipsoid of pulsation." The magnitudes and signs of the radial velocities which are induced along the principal axes of this ellipsoid by the passing second nebula approximately determine the character of the resulting pulsation in the nebula under consideration.

The probable existence of pulsation in many nebulae further complicates the interpretation of differences in the observed velocities V_s along the line of sight of various parts of a nebula. If we have pulsation, V_s differs from the value of V_t owing to rotation alone, by amounts which depend on the position of the tidal ellipsoid relative to the two vectors which define the line of sight and the axis of rotation. In two obvious, simple cases the functions $V_s(r)$ may graphically assume the forms represented by the two dotted curves (V_s and V_s') in Figure 1.

The possible existence of pulsation also suggests that we cannot, without additional knowledge, interpret ellipsoidal shapes of nebulae as being due to rotation alone. Values of the masses of nebulae which are derived on the assumption that elliptical nebulae are stellar systems in rotational equilibrium must therefore be viewed with suspicion.

Before embarking on any detailed analysis of spiral structures it will be advisable to obtain first qualitative information concerning the shape of the spiral arms in relation to the sense of rotation of the core. Promising results in this direction have already been secured by Dr. W. Baade of the Mount Wilson Observatory, who has found that the tips of the spiral arms of the Andromeda nebula are curved in the direction determined by the sense of rotation of the core, as pictured in Figure 2*b*. I am indebted to Dr. Baade for a private communication of this unpublished result.

Summing up, we may say that present data on internal rotations furnish, at best, minimum values M_{min} for the masses of the nebulae. Such minimum values are obtained if we assume that nebulae are stable systems whose components have velocities v inferior to the velocity of escape v_e from the system. Therefore

$$v_e \geqslant v_{\text{max}} \geqslant r_0\Omega_0 \,. \tag{9}$$

But

$$v_e = \left(\frac{2\Gamma M_0}{r_0}\right)^{1/2} , \tag{10}$$

where M_0, r_0, and Ω_0 are the mass, the radius, and the angular velocity of the core. Consequently, the mass M of the nebula is

$$M > M_0 \geqslant \frac{r_0^3\Omega_0^2}{2\Gamma} \,. \tag{11}$$

For example, in the case of NGC 4594 we have $r_0 \geqslant 4.3 \times 10^{21}$ cm and $r_0\Omega_0 \cong 4 \times 10^7$ cm/sec, which gives

$$M > 5 \times 10^{43} \text{ gr} = 2.5 \times 10^{10} M_\odot \,. \tag{12}$$

The data used are taken from the paper on NGC 4594 by F. G. Pease,[4] who, in 1916, measured the spectroscopic rotation of this nebula. Pease also was the first to point out that the central parts of nebulae rotate like solid bodies.

Since the determination of nebular masses is of considerable importance in modern astrophysics, three new methods for the solution of this problem are briefly outlined in the following pages. Two of these I have proposed already in previous communications.[5,6] The third, in a restricted form, has been applied thus far only to clusters of stars.[7] Its applicability to clusters of nebulae remains to be investigated. The purpose of this paper is to discuss the basic principles on which the new methods rest.

[4] *Proc. Nat. Acad.*, **2**, 517, 1916.

[5] F. Zwicky, *Helv. physica acta*, **6**, 110, 1933.

[6] F. Zwicky, *Phys. Rev.*, **51**, 290, 1937, and **51**, 679, 1937.

[7] H. von Zeipel, *Jubilaeumsnummer d. A. N.*, p. 33, 1921.

III. THE VIRIAL THEOREM APPLIED TO CLUSTERS OF NEBULAE

If the total masses of clusters of nebulae were known, the average masses of cluster nebulae could immediately be determined from counts of nebulae in these clusters, provided internebular material is of the same density inside and outside of clusters.

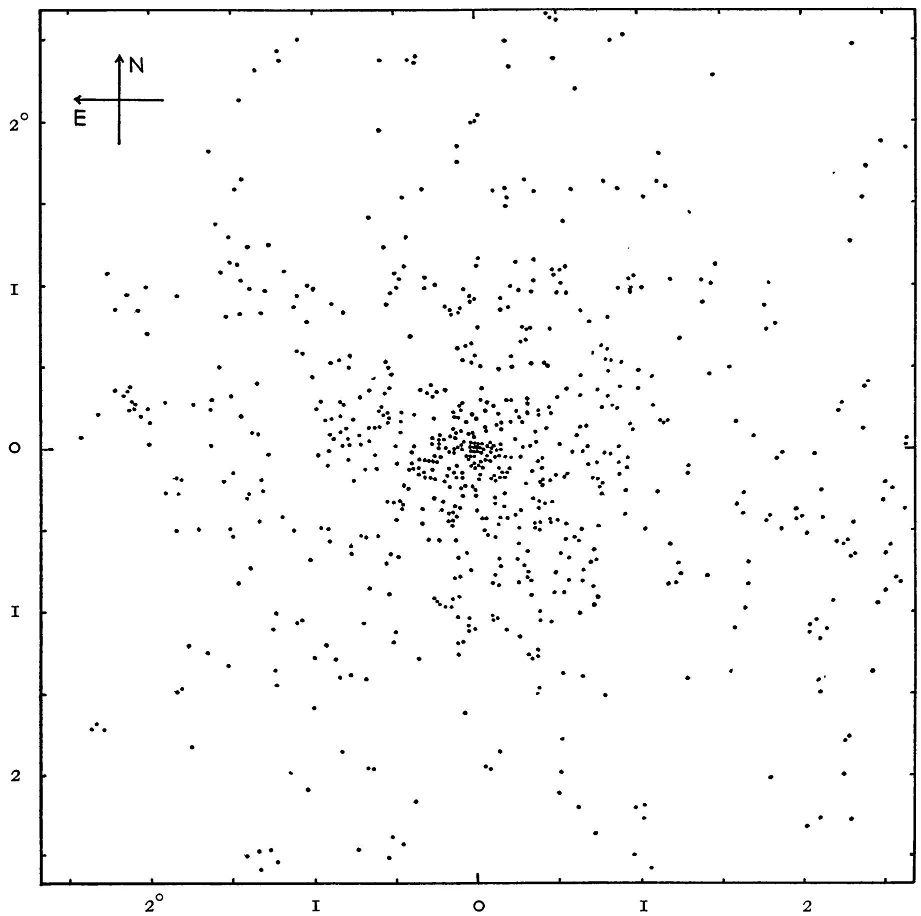

FIG. 3.—The Coma cluster of nebulae

As a first approximation, it is probably legitimate to assume that clusters of nebulae such as the Coma cluster (see Fig. 3) are mechanically stationary systems. With this assumption, the virial theorem of classical mechanics gives the total mass of a cluster in terms of the average square of the velocities of the individual nebulae which constitute this cluster.[5] But even if we drop the assumption that clusters represent stationary configurations, the virial theorem, in conjunction with certain additional data, allows us to draw important conclusions concerning the masses of nebulae, as will now be shown.

Suppose that the radius vector from a fixed point in the cluster to the nebula (σ) of mass M_σ is $\vec{r}_\sigma$. For the fixed point we conveniently chose the center of mass of the whole cluster. The fundamental law of motion of the nebula (σ) is

$$M_\sigma \frac{d^2\vec{r}_\sigma}{dt^2} = \vec{F}_\sigma , \tag{13}$$

where $\vec{F}_\sigma$ is the total force acting on M_σ. Scalar multiplication of this equation with $\vec{r}_\sigma$ gives

$$\tfrac{1}{2}\frac{d^2}{dt^2}(M_\sigma r_\sigma^2) = \vec{r}_\sigma \cdot \vec{F}_\sigma + M_\sigma\left(\frac{d\vec{r}_\sigma}{dt}\right)^2 . \tag{14}$$

Summation over all the nebulae of the cluster leads to

$$\tfrac{1}{2}\frac{d^2\Theta}{dt^2} = Vir + 2K_T , \tag{15}$$

where $\Theta = \sum_\sigma M_\sigma r_\sigma^2$ is the polar moment of inertia of the cluster, $Vir = \sum_\sigma \vec{r}_\sigma \cdot \vec{F}_\sigma$ is the virial of the cluster, and K_T is the sum of the kinetic energies of translation of the individual nebulae. If the cluster under consideration is stationary, its polar moment of inertia Θ fluctuates around a constant value Θ_0, such that the time average of its derivatives with respect to time is zero. Denoting time averages by a bar, we have in this case

$$\overline{Vir} = -2\overline{K_T} . \tag{16}$$

On the assumption that Newton's inverse square law accurately describes the gravitational interactions among nebulae, it follows that

$$Vir = E_p , \tag{17}$$

where

$$E_p = -\sum_{\sigma,\nu}\frac{\Gamma M_\sigma M_\nu}{r_{\sigma\nu}} \longleftrightarrow \sigma < \nu \tag{18}$$

is the total potential energy of the cluster due to the gravitational interactions of its member nebulae. Equation (16) thus takes on the well-known form

$$-\overline{E_p} = 2\overline{K_T} = \overline{\sum_\sigma M_\sigma v_\sigma^2} = \sum_\sigma M_\sigma \overline{v_\sigma^2}\,, \tag{19}$$

where v_σ is the velocity of the mass M_σ. In order to arrive at a quantitative estimate of the total mass $\mathcal{M}$ of a globular cluster of nebulae, we assume as a first approximation that these nebulae are, on the average, uniformly distributed inside a sphere of radius R. In this case

$$E_p = \frac{-3\Gamma\mathcal{M}^2}{5R}\,. \tag{20}$$

We may also write

$$\sum M_\sigma \overline{v_\sigma^2} = \mathcal{M}\,\overline{\overline{v^2}}\,, \tag{21}$$

where the double bar indicates a double average taken over time and over mass. Therefore, from (19), (20), and (21),

$$\mathcal{M} = \frac{5R\overline{\overline{v^2}}}{3\Gamma}\,. \tag{22}$$

This relation (22) can also be derived if we take the time average of equation (14) which holds for an individual nebula. The time average of the left side of (14) disappears if the mass M_σ is a member of a stationary system. Thus we have

$$\overline{vir_\sigma} = -2\overline{k_{\sigma T}} = -M_\sigma \overline{v_\sigma^2}\,, \tag{23}$$

where vir_σ is the virial of the nebula (σ) and $k_{\sigma T}$ is its kinetic energy. Now, in a sphere of mass $\mathcal{M}$ and radius R, the density ρ, if uniform, is equal to $\rho = 3\mathcal{M}/4\pi R^3$. The force $\vec{F}_\sigma(r)$ which acts on M_σ is therefore

$$\vec{F}_\sigma(r) = \frac{-\Gamma\mathcal{M}M_\sigma\vec{r}_\sigma}{R^3}\,, \tag{24}$$

and the virial

$$vir_\sigma = \vec{r}_\sigma \cdot \vec{F}_\sigma = \frac{-\Gamma \mathcal{M} M_\sigma r_\sigma^2}{R^3}, \tag{25}$$

which, combined with (23), results in

$$\frac{\Gamma \mathcal{M} \overline{r_\sigma^2}}{R^3} = \overline{v_\sigma^2}. \tag{26}$$

Since it has been assumed that all the nebulae combined produce a uniform distribution of matter throughout the sphere, the average nebula spends equal times in equal volumes, and we have

$$\overline{\overline{r^2}} = \frac{3}{4\pi R^3} \int_0^R r^2 \times 4\pi r^2 dr = \frac{3R^2}{5}, \tag{27}$$

where the double bar again designates a double average with respect to time and mass. Consequently,

$$\mathcal{M} = \frac{5R\overline{\overline{v^2}}}{3\Gamma}, \tag{28}$$

as before.

From the distribution of the brighter nebulae in the Coma cluster pictured in Figure 3 it is apparent that the assumption of uniform distribution is not fulfilled. But it is also evident that the actual potential energy E_p will have a value which, in order of magnitude, is correctly given by (20). Even if we crowded all the cluster nebulae into a sphere of radius $R/2$, the value of E_p would only be doubled. Also, we can increase E_p only very little if we assume that the M_σ's run through a wide range of values. For instance, if we assumed that practically the whole mass $\mathcal{M}$ of the cluster were concentrated in only two or three nebulae of mass $\mathcal{M}/2$ or $\mathcal{M}/3$, respectively, and that these masses had mutual distances as small as $R/10$, we should arrive at values for the potential energy:

$$E_p^{\mathrm{I}} = \frac{-2.5\Gamma\mathcal{M}^2}{R} \text{ and } E_p^{\mathrm{I}} = \frac{-\frac{10}{3}\Gamma\mathcal{M}^2}{R}. \tag{29}$$

These values are of the same order of magnitude as in (20). The following inequalities must therefore be considered as conservative estimates for the possible maximum values of the average kinetic energy and the minimum values of the total mass $\mathcal{M}$:

$$2\overline{K_T} = -\overline{E_p} < \frac{5\Gamma\mathcal{M}^2}{R} \tag{30}$$

and

$$\mathcal{M} > \frac{R\overline{\overline{v^2}}}{5\Gamma}\,. \tag{31}$$

We apply this relation to the Coma cluster of nebulae whose radius is of the order of 2×10^6 light-years. From the observational data we do not know directly the velocities v of the individual nebulae relative to the center of mass of the cluster. Only the velocity components v_s along the line of sight from the observer are known from the observed spectra of cluster nebulae. For a velocity distribution of spherical symmetry, however, we have

$$\overline{\overline{v^2}} = 3\overline{\overline{v_s^2}}\,. \tag{32}$$

Therefore

$$\mathcal{M} > \frac{3R\overline{\overline{v_s^2}}}{5\Gamma}\,. \tag{33}$$

From the observations of the Coma cluster so far available we have, approximately,[5]

$$\overline{\overline{v_s^2}} = 5 \times 10^{15}\text{cm}^2\ \text{sec}^{-2}\,. \tag{34}$$

This average has been calculated as an average of the velocity squares alone without assigning to them any mass weights, as actually should be done according to (21). It seems, however, as Sinclair Smith[8] has shown for the Virgo cluster, that the velocity dispersion for bright nebulae is about the same as that for faint nebulae. Assuming this to be true also for the Coma cluster, it follows that the

[8] *Ap. J.*, **83**, 499, 1936.

mass-weighted means of v^2 and the straight means are essentially the same. Furthermore, in calculating (34) we have used velocities which belong to the bright nebulae, since only these have been measured. If brightness can be taken as a qualitative indication of mass, the error in substituting (34) for (21) cannot be great. We must, nevertheless, remember that, strictly speaking, the determination of $\mathcal{M}$ by the virial theorem is subject to the difficulty of calculating $\overline{\overline{v^2}}$ through the application of an averaging process which involves the as yet unknown masses. The mass $\mathcal{M}$, as obtained from the virial theorem, can therefore be regarded as correct only in order of magnitude.

Combining (33) and (34), we find

$$\mathcal{M} > 9 \times 10^{46} \text{gr} . \tag{35}$$

The Coma cluster contains about one thousand nebulae. The average mass of one of these nebulae is therefore

$$\overline{M} > 9 \times 10^{43} \text{ gr} = 4.5 \times 10^{10} M_{\odot} . \tag{36}$$

Inasmuch as we have introduced at every step of our argument inequalities which tend to depress the final value of the mass $\mathcal{M}$, the foregoing value (36) should be considered as the lowest estimate for the average mass of nebulae in the Coma cluster. This result is somewhat unexpected, in view of the fact that the luminosity of an average nebula is equal to that of about 8.5×10^7 suns. According to (36), the conversion factor γ from luminosity to mass for nebulae in the Coma cluster would be of the order

$$\gamma = 500 , \tag{37}$$

as compared with about $\gamma' = 3$ for the local Kapteyn stellar system. This discrepancy is so great that a further analysis of the problem is in order. Parts of the following discussion were published several years ago, when the conclusion expressed in (36) was reached for the first time.[5]

We inquire first what happens if the cluster considered is not st-a tionary, in which case the virial theorem (16) must be replaced by one of the two inequalities

$$2\overline{K_T} + \overline{Vir} \gtrless 0 \,, \tag{38}$$

where the bars denote averages taken over time intervals which are comparable with the time it takes one nebula to traverse the whole system. The smaller sign needs no further consideration, since instead of resulting in equation (19) it leads to the inequality

$$-\overline{E_p} > 2\overline{K_T} \,. \tag{39}$$

This, in turn, means that the inequality (33) is further enhanced, and we arrive at a lowest value for the mass $\mathscr{M}$ greater even than (36).

The other alternative is

$$-\overline{E_p} < 2\overline{K_T} \,. \tag{40}$$

We may combine this inequality with the principle of conservation of total energy E of the whole cluster

$$\overline{K_T} + \overline{E_p} = E \,, \tag{41}$$

which, added to (40), leads to

$$E > -\overline{K_T} \,. \tag{42}$$

This means that the cluster is expanding. In particular, if $E = 0$ we have

$$\overline{E_p} = -\overline{K_T} \tag{43}$$

instead of $\overline{E_p} = -2\overline{K_T}$, for a stationary system. Equation (43) means that the cluster will ultimately just fly apart. In this case the mass $\mathscr{M}$ of the cluster still has a value of half that arrived at in (36), and the discrepancy between the conversion factors γ and γ' remains materially the same in order of magnitude.

If we wish to reduce the mass $\mathscr{M}$ still further so as to make approximately $\gamma = \gamma'$, we must put $E > 0$ and $K_T \gg -E_p$. By assuming this, however, we run into two serious difficulties. In the first place, it is difficult to understand why under these circumstances there are any great clusters of nebulae remaining in existence at all, since the formation of great clusters by purely geometrical chance is vanishingly small, as will be shown in another paper. In the second place, the cluster nebulae, after complete dispersion of a cluster would still possess velocities practically identical with their original velocities. The field nebulae in general, which under the assumed circumstances can hardly have a velocity distribution different from that of former cluster nebulae, should therefore have a dispersion in peculiar velocities comparable to that of cluster nebulae. The average range of peculiar velocities among field nebulae, however, seems to be of the order of 150 km/sec only. This observation, if correct, excludes values of the total energy E which are sufficiently greater than zero to reduce the discrepancy between γ and γ' to a satisfactory degree.[5] It will, nevertheless, be advisable to obtain more data on the velocities of both cluster nebulae and field nebulae in order to arrive at accurate values of the dispersion which characterizes the respective velocity distribution functions.

In addition it will be necessary to develop methods which allow us to determine the relative amounts of internebular material in clusters as well as in the general field.

It should also be noticed that the virial theorem as applied to clusters of nebulae provides for a test of the validity of the inverse square law of gravitational forces. This is of fundamental interest because of the enormous distances which separate the gravitating bodies whose motions are investigated. Since clusters of nebulae are the largest known aggregations of matter, the study of their mechanical behavior forms the last stepping-stone before we approach the investigation of the universe as a whole.

The result (36) taken at face value of course does not mean that the average masses of field nebulae must be as great as those of cluster nebulae. From the general principles discussed in a following section one would rather expect the heaviest nebulae to be favored in the process of clustering.

The distribution of nebulae in the Coma cluster, illustrated in Figure 3, rather suggests that stationary conditions prevail in this cluster. It is proposed, therefore, to study the Coma cluster in more detail. On the other hand, the virial theorem can hardly be used with much confidence in cases such as the Virgo cluster and the Pisces cluster.[9] These clusters are much more open and asymmetrical than the Coma cluster and their boundaries are thus far ill defined. Accurate values of the gravitational potentials in these clusters are difficult to determine.

In passing it should be noted that the mechanical conditions in clusters of nebulae are in some important respects different from the conditions in clusters of stars. During close encounters of stars only a minute part of their translational energy is transformed into internal energy of these stars, if the extremely rare cases of actual impacts are disregarded. Nebulae act differently. In the first place, close encounters and actual impacts in a cluster of nebulae must occur during time intervals which are not very long compared with the time of passage of one nebula through the entire system. Therefore, a considerable tendency exists toward equipartition of rotational and internal energy of nebulae with their translational energy. Star clusters in some ways are analogous to gas spheres built up of monatomic gases, whereas clusters of nebulae may be likened to gas spheres built up of polyatomic gases. This difference in internal characteristics may lead ultimately to serious consequences, as will be seen from the following line of reasoning.

We again start from the virial theorem (19), in the form in which it may be written for stationary gravitational systems, namely,

$$\overline{E_p} + 2\overline{K_T} = 0 . \tag{44}$$

Admitting that on close encounters of nebulae kinetic energy of translation K_T may be transformed into energy of rotation K_R and internal energy E_I of nebulae, we must replace the restricted form (41) of the energy principle by the more general equation

$$\overline{K_T} + \overline{K_R} + \overline{E_I} + \overline{E_p} = E . \tag{45}$$

Combining (44) and (45), we have

$$E = - \overline{K_T} + \overline{K_R} + \overline{E_I} . \tag{46}$$

[9] F. Zwicky, *Proc. Nat. Acad.*, May, 1937.

Encounters among nebulae tend to establish equipartition among translational, rotational, and internal energies of nebulae, analogous to the equipartition of energy among different degrees of freedom which is so well known in ordinary statistical mechanics. The mechanical conditions in a cluster of nebulae should converge toward a state for which

$$\overline{K_T} = \overline{K_R} . \quad \overline{E_I} = \lambda \overline{K_T} , \tag{47}$$

where $\lambda > 0$. For instance, $\lambda = 1$ if we include in E_I only the kinetic energy of the first three fundamental modes of pulsation of the whole system. The older the system becomes, the more of its many degrees of freedom may be expected to share in the equipartition of energy. Other happenings excluded, the value of λ would, therefore, increase almost indefinitely as the average value $\overline{E_I}$ of E_I is determined for time intervals of increasing length. In any case, admitting (47),

$$E = \lambda \overline{K_T} > 0 . \tag{48}$$

We are thus led to the uncomfortable conclusion that the total energy of a stationary cluster of nebulae should be positive. This is a contradiction in itself, since it means that on the assumption of stationary conditions in a cluster we have proved that the cluster really cannot be stationary but must ultimately fly apart. It would seem, therefore, that clusters of nebulae analogous to gravitational gas spheres[3] which are built up of polyatomic gases could never represent stationary configurations. In reality, however, condition (47) cannot be reached. In contradistinction to the rotational energy of polyatomic molecules at low temperatures, the rotational energy k_R of a nebula cannot become equal to its observed translational energy k_T. Long before the equipartition (47) among the various types of energy can be established by close encounters among nebulae, these nebulae will have been partially or completely disrupted. We are here confronted with processes which are analogous to the dissociation of polyatomic molecules when their average kinetic energy of translation—that is, the temperature of the gas—becomes too high.

The preceding considerations open up interesting new vistas on the change in time of nebular types in clusters where close encounters are much more frequent than among field nebulae. In the first place, the most compact and most massive nebulae are presumably the least vulnerable to disruption. These nebulae, therefore, are destined to survive the longest. This effect of selective elimination of nebular types may, in part, be responsible for the difference in the representation of types among cluster nebulae as compared with field nebulae. In the second place, we should expect a considerable number of stars, as well as matter in dispersed form from disrupted nebulae, to be scattered through the internebular spaces within clusters. Sufficiently large amounts of internebular matter in clusters might seriously change our estimate (36) of the average value of nebular masses as derived from the preceding application of the virial theorem to clusters of nebulae. It is therefore the intention to undertake a series of observations which may throw some light on the problem of the density of internebular matter in clusters, as compared with the density of matter in the general field. Until such observations have been made it will be well to keep in mind that, although the determination of average nebular masses from the virial theorem may be viewed with considerable confidence, this method is not entirely free from objections which have not yet been satisfactorily dealt with.

In principle the virial theorem may also be applied to describe the mechanical conditions in an individual nebula. Actually a direct application is difficult, since it is not possible to measure separately, as in the case of a cluster of nebulae, the velocities of the individual units of mass which constitute a nebula. The average square velocity (21) might be derived from the shape of the spectral lines in the light from nebulae. Unfortunately, the practical determination of such shapes is at present exceedingly difficult, if not impossible. In addition the spectral lines in the light of nebulae are doubtless of complex origin, and the interpretation even of well-known shapes of lines is by no means an easy task.

IV. NEBULAE AS GRAVITATIONAL LENSES

As I have shown previously,[6] the probability of the overlapping of images of nebulae is considerable. The gravitational fields of a number of "foreground" nebulae may therefore be expected to deflect the light coming to us from certain background nebulae. The observation of such gravitational lens effects promises to furnish us with the simplest and most accurate determination of nebular masses. No thorough search for these effects has as yet been undertaken. It would seem, perhaps, that if the masses of field nebulae were, on the average, as great as the masses of cluster nebulae obtained in section iii, gravitational lens effects among nebulae should have been long since discovered. Until many plates of rich nebular fields taken under excellent conditions of seeing have been carefully examined it would be dangerous, however, to draw any definite conclusions.

The mathematical analysis of the formation of images of distant nebulae through the action of the gravitational fields of nearer nebulae will be given in detail in an article to be published in the *Helvetica physica acta.*

V. STATISTICAL DISTRIBUTION IN SPACE OF DIFFERENT TYPES OF NEBULAE

It will be shown elsewhere that the number of clusters of nebulae actually observed is far greater than the number that might be expected for a random distribution of non-interacting objects. This tendency of nebulae toward clustering is no doubt due to the action of gravitational forces.

By a bold extrapolation of well-known results of ordinary statistical mechanics we adopt the following working hypothesis as a tentative basis for the interpretation of future observations on the clustering of nebulae.

BASIC PRINCIPLES

1. The system of extragalactic nebulae throughout the known parts of the universe forms a statistically stationary system.

2. Every constellation of nebulae is to be endowed with a probability weight $f(\epsilon)$ which is a function of the total energy ϵ of this constellation. Quantitatively the probability P of the occurrence of a certain configuration of nebulae is assumed to be of the type

$$P = A\left(\frac{V}{V_0}\right) f\left(\frac{\epsilon}{\epsilon_K}\right) . \qquad (49)$$

Here V is the volume occupied by the configuration or cluster considered, V_0 is the volume to be allotted, on the average, to any individual nebula in the known parts of the universe, and ϵ is the total energy of the cluster in question, while $\overline{\epsilon_K}$ will probably be found to be proportional to the average kinetic energy of individual nebulae. The function $A(V/V_0)$ can be determined a priori. On the other hand, $f(\epsilon/\overline{\epsilon_K})$ presumably will be found to be a monotonously decreasing function in $\epsilon/\overline{\epsilon_K}$, analogous in type to a Boltzmann factor

$$F = \text{const} \times e^{-\epsilon/\overline{\epsilon}_K} . \tag{50}$$

Assuming the basic principles stated in the preceding to be correct, we may draw the following hypothetical conclusions:

a) The clustering of nebulae is favored by high values of f and is partially checked by low values of the a priori probability A.

b) If, as would appear to be certain, nebulae are not all of the same mass, nebulae of high mass are favored in the process of clustering, since they contribute most to produce high values of the weight function f.

c) As a consequence of *b*, we should expect that the frequency with which different types of nebulae occur will not be the same among field nebulae and among cluster nebulae. In other words, clustering is a process which tends to segregate certain types of nebulae from the remaining types. This may contribute toward the correct interpretation of the well-known fact that cluster nebulae are preponderantly of the globular and elliptical types, whereas field nebulae are mostly spirals. From the arguments put forth in the preceding section as well as in section iii it follows that it is not necessary as yet to call on evolutionary processes to explain why the representation of nebular types in clusters differs from that in the general field. Here, as in the interpretation of other astronomical phenomena, the idea of evolution may have been called upon prematurely. It cannot be overemphasized in this connection that systematic and irreversible evolutionary changes in the domain of astronomy have thus far in no case been definitely established.

d) If cluster nebulae, on the average, are really more massive than field nebulae, the conclusion suggests itself that globular nebulae may, somewhat unexpectedly, be among the most massive systems. It will be of great interest to check this inference by a search for gravitational lens effects among globular nebulae.

The preceding considerations point toward the possibility of an entirely new approach in the study of masses of nebulae. We may argue somewhat as follows:

The function $A(V/V_0)$, as said before, can be obtained from the theory of probabilities applied to random distributions in space of non-interacting objects. The function A, therefore, is known a priori. The function f may be determined from counts of types of clusters of nebulae (single, double, triple, quadruple nebulae, etc.) in given volumes of space. Such counts will give directly the values of P characteristic for different clusters. The numerical values of the function f follow from $f = P/A$. In order to determine the masses of individual nebulae, we must express the argument $\epsilon/\overline{\epsilon_K}$ of the function f in terms of these masses and then seek to correlate each definite argument with one numerical value of f. To solve this problem of the functional form of f we may proceed as follows:

We first segregate the nebulae into classes of types $T_1, T_2, T_3, \ldots . T_n$. For these types the usual ones, $E_0, E_1, \ldots . E_7, S_a, S_b, S_c$, etc., may, for instance, be chosen. As a first approximation we assume tentatively that the mass $M_\sigma(L)$ of a nebula of a given type T_σ is a function of its luminosity L alone. The argument $\epsilon/\overline{\epsilon_K}$ of the function f may then be formulated mathematically in terms of M_σ, $r_{\sigma\nu}$ and v_σ, where the M_σ are the unknown masses of the various types (T_σ) of nebulae in the cluster and where the velocities v_σ of these nebulae, as well as their mutual distances $r_{\sigma\nu}$, are known from observation. Since the numerical values of f are already known, the functional dependence of these values on the arguments $\epsilon/\overline{\epsilon_K}$ can then be determined by a purely mathematical procedure. Once the form of the function $f(\epsilon/\overline{\epsilon_K})$ is known, the unknown masses follow from our knowledge of $\epsilon/\overline{\epsilon_K}$ expressed in terms of M_σ, v_σ, $r_{\sigma\nu}$. It should be noted, however, that the method just described gives only relative masses M_σ/M_0, measured, for instance, with the mass M_0 of the type T_0 taken as the arbitrary unit. Only if the absolute value of $\overline{\epsilon_K}$ is known or if we have independent knowledge of M_0 can we derive the absolute values M_σ from the statistics of nebular distribution.

Needless to say, this program, if it can be carried out, provides the most powerful method of determining the masses of all types of nebulae. In addition, it also enables us to determine the statistical weights f. The practical application of this method necessitates a great amount of observational work. In view of this fact, it will perhaps be advantageous to apply the preceding program first in a restricted form by a consideration of the distribution of various types of nebulae in one individual great cluster. The procedure to be applied in this case is analogous to that used successfully by H. von Zeipel[7] in his determination of the masses of different types of stars in certain clusters of stars.

Since it is intended to carry out the investigation just mentioned on the Coma cluster, a few preliminary remarks concerning the distribution of nebulae in this cluster are here given.

VI. THE COMA CLUSTER OF NEBULAE

According to Hubble and Humason,[10] the Coma cluster of nebulae "consists of about 800 nebulae scattered over an area roughly 1°7 in diameter. . . . . Photographic magnitudes range from about 14.1 to 19.5, with 17.0 as the most frequent value." The distance of the cluster is about 13.8 million parsecs.

In order to get some preliminary data on the distribution of nebulae in the Coma cluster, photographs of this cluster were taken on Mount Palomar with the new 18-inch Schmidt-type telescope of the California Institute of Technology. The faintest nebulae which on limiting exposures (30–60 min) with this telescope can still be clearly distinguished from stars have an apparent magnitude close to 16.5. In Figure 3 dots represent nebulae which can be distinguished on 30-minute exposures on panchromatic films. To avoid crowding of points near the center of the cluster, not all the nebulae in this region are marked which can be seen on the original photographs. The counts given in Table 1 at different distances r from the

[10] E. Hubble and M. L. Humason, *Ap. J.*, **74**, 131, 1931; see also the descriptions of the Coma cluster by M. Wolf, *Heidelberg Pub.*, **1**, 125, 1902, and H. Shapley, *Harvard Bull.*, No. 896, 1934.

center of the Coma cluster include, however, all the nebulae which I have been able to identify on half-hour exposures.

The nebula NGC 4874 (α 12^h56^m, δ $28°20'$, 1930), which lies at (0°, 0°) in Figure 3, was taken to be the approximate center of the cluster, no effort being made to determine a mathematically accurate central point. Concentric circles were then drawn around the adopted center, with radii $r = nr_0$, where n is a whole number running from $n = 1$ to $n = 32$, and $r_0 \cong 5$ minutes of arc. The unit of area to

TABLE 1

COUNTS OF NEBULAE IN THE COMA CLUSTER

r/r_0	n_r	$\log_{10}n_r$	r/r_0	n_r	$\log_{10}n_r$
1/5	50	1.699	12	1.61	0.207
1/3	63	1.799	13	1.20	.079
1/2	52	1.716	14	1.44	.158
1	31	1.491	15	0.66	− .181
			16	0.48	− .319
2	14	1.146	17	0.60	− .222
3	10.8	1.033	18	0.48	− .319
4	6.33	0.801	19	0.43	− .367
5	4.22	0.625	20	0.64	− .194
6	4.10	0.613	22	0.26	− .585
7	1.85	0.267	24	0.36	− .444
8	2.73	0.436	26	0.27	− .569
9	1.76	0.245	28	0.16	− .796
10	2.16	0.334	30	0.25	− .602
11	1.76	0.245	32	0.15	−0.824

which all counts are reduced is $s = \pi r_0^2$, or about 1/46.4 sq. deg. The numbers of nebulae per unit area, n_r, are averages for the ringlike areas which lie between $r = nr_0$ and $r = (n + 1)r_0$. The first four figures in Table 1, however, are averages for the full circles the radii of which are $r = r_0/5$, $r_0/3$, $r_0/2$, and r_0, respectively. The corresponding numbers N_r of nebulae per square degree are $N_r = 46.4\,n_r$.

In Figure 4 values of $\log_{10} n_r$ are plotted against values of r. The full curve is drawn only approximately and does not correspond to any definite mathematical function. From the general character of this curve it is seen immediately that the Coma cluster extends to much greater distances than was originally assumed by Hubble and Humason.[10] At the edge of a circle the diameter of which is 4°.5 instead of only 1°.7, the average number of nebulae per unit area is still higher than the corresponding number in the surrounding general field. Since our counts include only the brighter nebulae, it is to be expected that counts made with more powerful telescopes will enable us to follow the extensions of the Coma cluster still farther into the general field.

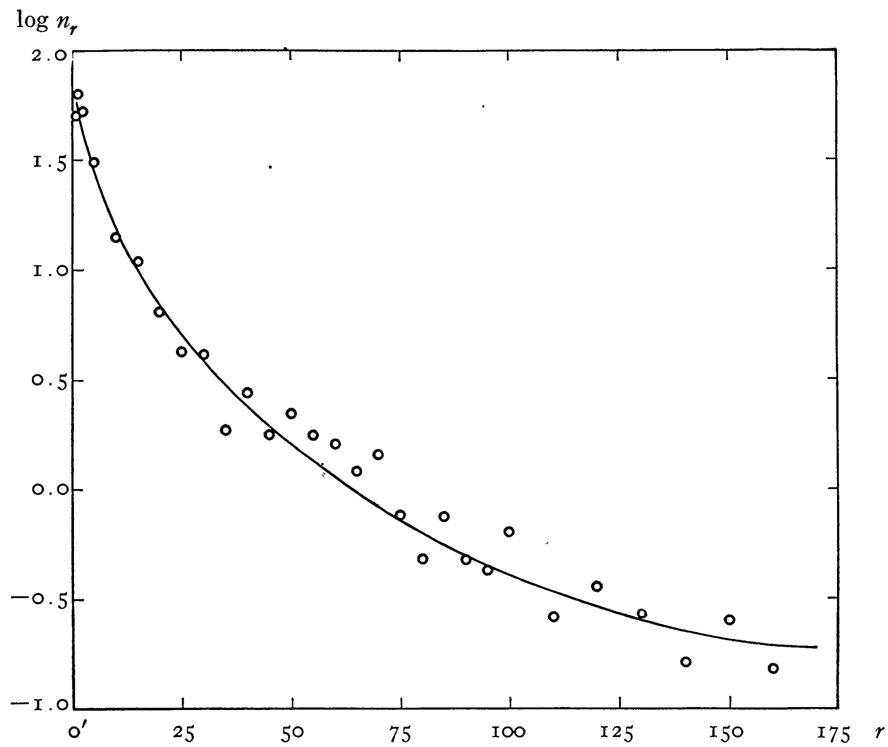

FIG. 4.—Counts of nebulae in the Coma cluster

The high central condensation, the very gradual decrease of the number of nebulae per unit volume at great distances from the center of a cluster, and the great extension of this cluster become here apparent for the first time. It is quite as we should expect from the considerations of section v. According to these considerations, a cluster of nebulae analogous to an isothermal gravitational gas sphere may in some cases be expected to extend indefinitely far into space, until its extension is stopped through the formation of independent clusters in the regions surrounding it.

The actual shape of the distribution curve in Figure 4 is also of great interest. We notice at once the great similarity of this curve to the luminosity-curves of elliptical nebulae derived by Hubble.[11] According to him, the distribution of the intrinsic luminosity in globular nebulae corresponds very closely to the distribution of mass density in isothermal gas spheres as computed by R. Emden.[3] The same is approximately true for the distribution of the brighter nebulae in the Coma cluster. This result also checks the general conclusions drawn from the basic principles which, according to the discussion in section v, determine the stationary configurations of clusters of nebulae.

The total number $\mathcal{N}$ of nebulae in the Coma cluster that can be identified on photographs taken with the 18-inch Schmidt telescope is obtained as follows: The curve in Figure 4 apparently has a horizontal asymptote which corresponds to a value of n_r not higher than $n_\infty = 0.159$. Thus, the corresponding number N of nebulae in the general field should be $N = 46.4 \times n_\infty = 7.38$. The total number of nebulae counted to the distance $r = 2°40'$ from the center is 834. From this we must subtract $22.3 \times 7.38 = 165$ nebulae, since the area in question covers 22.3 sq. deg. The Coma cluster therefore comprises a number $\mathcal{N}$ of nebulae the brightnesses of which are greater than $m = 16.5$:

$$\mathcal{N}_{16.5} = 670 . \tag{51}$$

Since the most frequent apparent magnitude is about 17, we conclude that $\mathcal{N}_{16.5}$ is less than half the total number $\mathcal{N}$ of nebulae incorporated in the Coma cluster. This number must be at least equal to $\mathcal{N} = 1500$, and it may be even greater.

Finally, a word with respect to the limiting magnitude $m = 16.5$ which we have used in the preceding discussion. According to Hubble,[2] the number N_m of nebulae per square degree which are brighter than m near the galactic pole is given by

$$\log N_m = 0.6m - 9.05 . \tag{52}$$

Since the Coma cluster lies within a few degrees of the galactic pole, we may use relation (52) directly without introducing any correction for partial obscuration. Our counts of nebulae lead to an aver-

[11] E. Hubble, *Ap. J.*, 71, 131, 1930.

age number of 7.38 nebulae in the general field surrounding the Coma cluster. Inserting $N_m = 7.38$ into the equation (52), we therefore find that $m = 16.6$ represents approximately the limiting magnitude at which it is still possible to distinguish images of average nebulae from those of stars on photographs taken with the 18-inch Schmidt telescope.

VII. COMPARISON OF THE THREE METHODS

Each of the three new methods for the determination of masses of nebulae which have been described makes use of a different fundamental principle of physics. Thus, method iii is based on the virial theorem of classical mechanics; method iv takes advantage of the bending of light in gravitational fields; and method v is developed from considerations analogous to those which result in Boltzmann's principle in ordinary statistical mechanics. Applied simultaneously, these three methods promise to supplement one another and to make possible the execution of exacting tests to the results obtained.

Method iii can be applied advantageously only to clusters. Its application calls for the observation of radial velocities of cluster nebulae. The absolute dimensions of the cluster investigated also must be known.

Method iv involves the observation of gravitational lens effects. Measurements of deflecting angles combined with data on the absolute distance of the "lens nebula" from the observer suffice to determine the mass of the lens nebula. The chances for the successful application of this method grow rapidly with the size of the available telescopes. Since method iii gives only the average masses of cluster nebulae and method v furnishes only the ratios between the masses of different types of nebulae, much depends on whether or not a single image of a nebula, modified through the gravitational field of another nebula, can be found. A single good case of this kind would, so to speak, provide us with the fixed point of Archimedes in our attempt to explore the physical characteristics of nebulae.

Method v is the most powerful of all, since it enables us in principle to find the masses of all types of nebulae, provided the absolute mass of a single type of nebula is known or that we have some independent way of finding a sufficiently accurate value of $\overline{\epsilon_K}$. Method v also results automatically in the knowledge of the statistical weight functions *f* which govern the distribution of nebulae. The knowledge of these functions is of interest for two reasons:

1. The weight functions derived from direct observations may be compared with those to be expected theoretically, for different "models" of the universe. Through such a comparison it should be possible to decide whether the universe as a whole is in thermodynamic equilibrium[12, 13] or is continually changing.

2. It will be of particular interest, as proposed previously,[14] to investigate the probability P of the occurrence of clusters of nebulae in our "immediate" extragalactic neighborhood, as well as at great distances. If the universe is, for instance, expanding, we should expect P to be different at different distances from the observer. The fact that a great cluster of nebulae, such as the Coma cluster, seems to represent a statistically stationary configuration suggests that a short time scale with 10^9 years as the characteristic age of the universe is hardly adequate. Considerably longer time intervals would seem to be necessary to insure the formation of a stationary distribution of nebulae in great clusters. A detailed analysis of the problem of the time scale, however, must be postponed until the distribution of nebulae in a greater number of clusters has been investigated.

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[12] P. S. Epstein, *Commentary on the Scientific Writing of J. W. Gibbs* (New Haven: Yale University Press, 1936), **2**, 104.

[13] F. Zwicky, *Proc. Nat. Acad.*, **14**, 592, 1928.

[14] F. Zwicky, *Phys. Rev.*, **48**, 802, 1935.