Basic Radar 3.1: Probability Theory for Incoherent Scatter Radar

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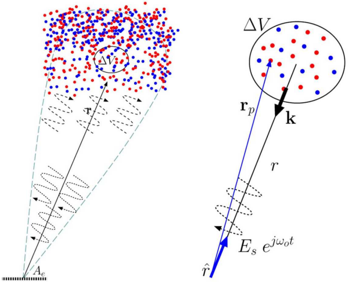
July, 2020

The Need for Statistical Descriptions of ISR Signals

If I knew the positions of every single electron in the scattering volume, I would know the received voltage exactly:



Exact expression for scattered electric field as a superposition of Thompson scatterers:

$$E_s = -\frac{r_e}{r} E_0 \sum_{p=1}^{N_0 \Delta V} e^{j\mathbf{k}\cdot\mathbf{r}_p}$$

ISR theory predicts statistical aspects of the scattered signal:

Scattered Power: $\left\langle |E_s|^2 \right\rangle$ Autocorrelation Function: $\langle E_s(t) E_s^*(t-\tau) \rangle$

These statistical properties are functions of macroscopic properties of the plasma: N_e, T_e, T_i, u_{los}.

Random Variables

A **random variable** is a variable whose numerical value depends on the outcome of a probabilistic phenomenon.

Probability Density Function:

$$P(x_1 < X < x_2) = \int_{x_1}^{x_2} p_X(x)\, dx$$

Expected Values:

$$E\{g(X)\} = \int_{-\infty}^{\infty} g(x) p_X(x)\, dx$$

Mean:

$$Mean\{X\} = E\{X\} = \bar{X}$$

Variance:

$$Var\{X\} = E\left\{(X - E\{X\})^2\right\} = E\{X^2\} - (E\{X\})^2$$

Collections of Random Variables

Multiple RVs must be described by joint-PDFs:

$$P\left(x_0 < X < x_1 \cup y_0 < Y < y_1\right) = \int_{x_0}^{x_1} \int_{y_0}^{y_1} p_{XY}\left(x, y\right)\, dydx$$

If X and Y are **independent**:

$$p_{XY}(x,y) = p_X(x)p_Y(y) \qquad p_{X|Y}(x|y) = p_X(x)$$

Relationships between RVs are defined through covariances:

$$Cov\left\{X, Y\right\} = E\left\{\left(X - E\{X\}\right)\left(Y - E\{Y\}\right)\right\}$$

Uncorrelated RVs have $Cov\{X, Y\} = 0$
Independent RVs are uncorrelated, but uncorrelated RVs are not necessarily independent.

Random Vectors

Column vector of random variables

$$\mathsf{X} = \begin{pmatrix} X_0 \\ X_1 \\ \vdots \\ X_{N-1} \end{pmatrix}$$

Covariance matrix of a random vector

$$K_{\mathsf{X}} = \operatorname{Cov}\{\mathsf{X}\} = E\{\left(\mathsf{X} - \bar{\mathsf{X}}\right)\left(\mathsf{X} - \bar{\mathsf{X}}\right)^T\}$$

$$= \begin{pmatrix} \operatorname{Var}\{X_0\} & \operatorname{Cov}\{X_0, X_1\} & \cdots & \operatorname{Cov}\{X_0, X_{N-1}\} \\ \operatorname{Cov}\{X_1, X_0\} & \operatorname{Var}\{X_1\} & \cdots & \operatorname{Cov}\{X_1, X_{N-1}\} \\ \vdots & \vdots & \ddots & \vdots \\ \operatorname{Cov}\{X_{N-1}, X_0\} & \operatorname{Cov}\{X_{N-1}, X_1\} & \cdots & \operatorname{Var}\{X_{N-1}\} \end{pmatrix}$$

Cross-covariance of two random vectors

$$K_{\mathsf{XY}} = \operatorname{Cov}\{\mathsf{X}, \mathsf{Y}\} = E\{\left(\mathsf{X} - \bar{\mathsf{X}}\right)\left(\mathsf{Y} - \bar{\mathsf{Y}}\right)^T\}$$

Gaussian Distribution

A Gaussian random variable X has the following probability density function (Normal Distribution):

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{x-\mu}{2\sigma^2}\right\}$$

$$E\{X\} = \mu \qquad Var\{X\} = \sigma^2$$

$$E\left\{(X-\mu)^4\right\} = 3\sigma^4$$

A jointly-Gaussian vector of random variables $\mathrm{X} = [X_0, X_1, X_2, \cdots, X_{N-1}]^T$ has the joint pdf:

$$p(\mathrm{x}) = \frac{1}{(2\pi)^{\frac{N}{2}} |K|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}[\mathrm{x}-\mu]^T K^{-1} [\mathrm{x}-\mu]\right\}$$

$$E\{\mathrm{X}\} = \mu$$

$$Cov\{\mathrm{X}\} = E\left\{[\mathrm{X}-\mu][\mathrm{X}-\mu]^T\right\} = K$$

Central Limit Theorem

Given a set of finite-variance, independent and identically distributed RV, $[X_0, X_1, \cdots, X_{K-1}]$, the distribution function of the average:

$$\hat{X} = \frac{1}{K} \sum_{n=0}^{K-1} X_n$$

will asymptotically approach a Gaussian distribution as K increases.

$$E\left\{\hat{X}\right\} = E\left\{X_n\right\} \qquad Var\left\{\hat{X}\right\} = \frac{1}{K} Var\left\{X_n\right\}$$

This is an amazingly useful theorem:

- Only the mean and variances of the intermediate quantities need to be calculated to predict the distribution of the final averaged result.
- Distribution functions of intermediate quantities do not need to be calculated in detail since the final averaged result will just be Gaussian.

Properties of Jointly Gaussian Random Variables

- Linear combinations:

$$Z = \alpha X + \beta Y + \gamma \quad E\{Z\} = \alpha E\{X\} + \beta E\{Y\} + \gamma$$
$$Var\{Z\} = \alpha^2 Var\{X\} + \beta^2 Var\{Y\} + 2\alpha\beta Cov\{X, Y\}$$

- Matrix generalization:
  $\mathrm{Y} = \mathrm{A}\mathrm{X} + \mathrm{b} \quad E\{\mathrm{Y}\} = \mathrm{A}E\{\mathrm{X}\} + \mathrm{b} \quad Cov\{\mathrm{Y}\} = \mathrm{A}Cov\{\mathrm{X}\}\mathrm{A}^T$

Special cases for zero mean random variables:

- Odd moments are zero:
  $E\{V_1\} = E\{V_1 V_2 V_3\} = E\{V_1 V_2 V_3 V_4 V_5\} = \cdots = 0$
- Fourth moment theorem: $E\{V_1 V_2 V_3 V_4\} =$
  $E\{V_1 V_2\} E\{V_3 V_4\} + E\{V_1 V_3\} E\{V_2 V_4\} + E\{V_1 V_4\} E\{V_2 V_3\}$
- General even moment theorem (Isserlis' Theorem)
  $E\{V_1 V_2 \cdots V_{2n-1} V_{2n}\} = \sum \prod E\{V_i V_j\}$

Complex-valued Random Variables

A complex-valued random variable X can be described by

$$\text{Mean:}\bar{X} = E\left\{X\right\}$$
$$\text{Covariance:}K_X = E\left\{\left(X - \bar{X}\right)\left(X - \bar{X}\right)^*\right\}$$
$$\text{Pseudo-Covariance:}J_X = E\left\{\left(X - \bar{X}\right)\left(X - \bar{X}\right)\right\}$$

A vector of complex-valued random variables X can be described by

$$\text{Mean:}\bar{\mathrm{X}} = E\left\{\mathrm{X}\right\}$$
$$\text{Covariance:}K_X = E\left\{\left(\mathrm{X} - \bar{\mathrm{X}}\right)\left(\mathrm{X} - \bar{\mathrm{X}}\right)^H\right\}$$
$$\text{Pseudo-Covariance:}J_X = E\left\{\left(\mathrm{X} - \bar{\mathrm{X}}\right)\left(\mathrm{X} - \bar{\mathrm{X}}\right)^T\right\}$$

Where $(\cdot)^H$ means Hermitian transpose (i.e. with a complex conjugate) and $(\cdot)^T$ means ordinary transpose.

Relationship to Real Random Variables

A vector of N complex-valued random variables can be written as two vectors of real-valued random variables representing the real and imaginary parts.

$$\mathsf{X} = \mathsf{X}_R + j\mathsf{X}_I$$

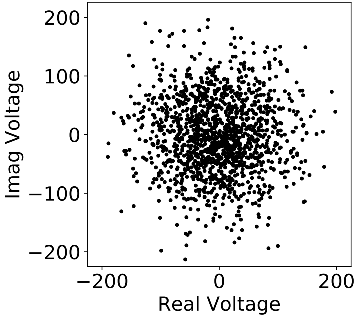
The covariance and cross-covariance matrices of these real vectors are related to the covariance and pseudo-covariance of the complex vector by

$$\begin{aligned}
\mathrm{Cov}\left\{\mathsf{X}_R\right\} &= \frac{1}{2}\Re K_X + \frac{1}{2}\Re J_X \\
\mathrm{Cov}\left\{\mathsf{X}_I, \mathsf{X}_R\right\} &= \frac{1}{2}\Im K_X + \frac{1}{2}\Im J_X \\
\mathrm{Cov}\left\{\mathsf{X}_R, \mathsf{X}_I\right\} &= -\frac{1}{2}\Im K_X + \frac{1}{2}\Im J_X \\
\mathrm{Cov}\left\{\mathsf{X}_I\right\} &= \frac{1}{2}\Re K_X - \frac{1}{2}\Re J_X
\end{aligned}$$

Special Properties of ISR Voltages

ISR Voltages will always be:

- Gaussian
- Zero Mean: $(E\{V\} = 0)$
- Finite power: $(E\{VV^*\} < \infty)$
- Random Phase:
  - Zero Pseudo-variance $(E\{VV\} = 0)$
  - $\mathrm{Cov}\{\Re V, \Im V\} = 0$
- Collections of ISR voltages will always have zero pseudo-covariance with each other.



With these properties, the complex-valued covariances between ISR voltages tells us everything we could want to know.

Describing ISR Voltages

ISR signals are complex valued, zero mean, and random phase.

$$V = V_R + jV_I \qquad E\left\{V_R\right\} = E\left\{V_I\right\} = 0$$

$$E\left\{VV^*\right\} = \sigma^2 \qquad E\left\{V_R V_I\right\} = 0 \qquad \mathrm{Cov}\left\{\begin{pmatrix} V_R \\ V_I \end{pmatrix}\right\} = \frac{1}{2}\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}$$

When we talk about correlations between ISR signals

$$E\left\{V_1 V_1^*\right\} = \sigma_1^2 \quad E\left\{V_2 V_2^*\right\} = \sigma_2^2$$

$$E\left\{V_1 V_2^*\right\} = \rho = \rho_R + j\rho_I$$

What we really mean is

$$V_1 = V_{1R} + jV_{1I} \qquad V_2 = V_{2R} + jV_{2I}$$

$$\mathrm{Cov}\left\{\begin{pmatrix} V_{1R} \\ V_{1I} \\ V_{2R} \\ V_{2I} \end{pmatrix}\right\} = \frac{1}{2}\begin{pmatrix} \sigma_1^2 & 0 & \rho_R & -\rho_I \\ 0 & \sigma_1^2 & \rho_I & \rho_R \\ \rho_R & \rho_I & \sigma_2^2 & 0 \\ -\rho_I & \rho_R & 0 & \sigma_2^2 \end{pmatrix}$$

Probability for ISR Summary

- The theory of ISR is a probabilistic theory for the statistical properties of the received voltages.
- ISR voltages are Gaussian, zero mean, random phase, complex-valued random variables.
- All the information we want is in the variances (power) and covariances between voltages.