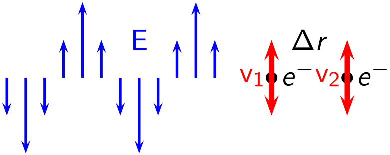
ISR Theory 2: Bragg Scatter

Roger H. Varney

[1]Center for Geospace Studies
SRI International

July, 2020

Scatter from Two Electrons



Incident on first electron:

$$E_1 = E_0 e^{j\omega t}$$

Scattered from first electron:

$$E_{s1} = -\frac{r_e}{r} E_1 e^{-jk_0 r}$$
$$= -\frac{r_e}{r} E_0 e^{j\omega t - jk_0 r}$$

Incident on second electron:

$$E_2 = E_0 e^{j\omega t - jk_0 \Delta r}$$

Scattered from second electron:

$$E_{s2} = -\frac{r_e}{r + \Delta r} E_2 e^{-jk_0 (r + \Delta r)}$$
$$= -\frac{r_e}{r} E_0 e^{j\omega t - jk_0 r - j2k_0 \Delta r}$$

In the far field $\frac{1}{r+\Delta r} \approx \frac{1}{r}$, so the sum of the fields is

$$E_{s1} + E_{s2} = -\frac{r_e}{r} E_0 e^{j\omega t - jk_0 r} \left(1 + e^{-j2k_0 \Delta r}\right)$$

Bragg Wavelength

For scatter from two electrons

$$|E_{s1} + E_{s2}|^2 \propto \left|1 + e^{-j2k_0\Delta r}\right|^2 = 4\cos^2(k_0\Delta r)$$

- If $\Delta r = \frac{\lambda}{2}$, $k_0\Delta r = \pi$, and the factor is 4 (perfect constructive interference)
- If $\Delta r = \frac{\lambda}{4}$, $k_0\Delta r = \frac{\pi}{2}$, and the factor is 0 (perfect destructive intereference)
- If Δr is a random number, the expected value of the factor is 2.

The Bragg wavelength $\lambda_b = \frac{\lambda_0}{2}$ is the preferred spacing where the scatter adds constructively.
Define the Bragg wavenumber (for backscatter) as $k_b = \frac{2\pi}{\lambda_b} = \frac{4\pi}{\lambda_0} = 2k_0$.

Scatter from Many Electrons

$$E_s = -\frac{r_e}{r} E_0 e^{j\omega t - jkr} \left(\sum_{p=0}^{N-1} e^{-j2\mathrm{k}_0 \cdot \Delta \mathrm{r}_p} \right)$$
$$= -\frac{r_e}{r} E_0 e^{j\omega t - jk_0 r} \int n_e(\mathrm{r})\, e^{-j2\mathrm{k}_0 \cdot \Delta \mathrm{r}_p} d^3 r$$

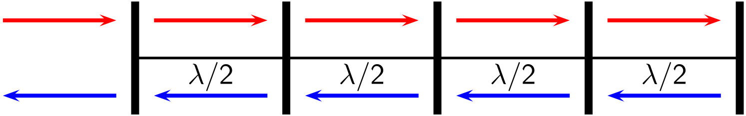
where the microscopic electron density is

$$n_e(\mathrm{r}) \equiv \sum_{p=0}^{N-1} \delta\left(\mathrm{r} - \Delta \mathrm{r}_p\right)$$

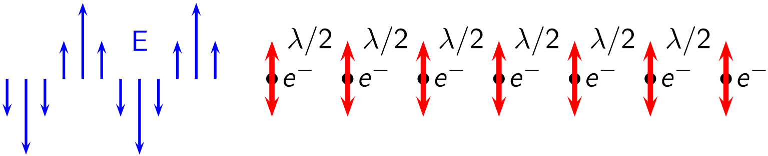
This looks like a spatial Fourier transform evaluated at the Bragg wavenumber $\mathrm{k}_b = 2\mathrm{k}_0$.
The scatter is most sensitive to density structures at the Bragg wavelength.

Distributed Bragg Reflectors

Stack of reflecting structures



Stack of electrons



Coherent vs Incoherent Scatter

- **Coherent Scatter**: If the plasma is unstable and full of irregularities at the Bragg wavelength, lots of constructive interference will occur, and the radar will receive lots of signal.
- **Incoherent Scatter**: The plasma is disorganized:

$$\left| \sum_{p=0}^{N-1} e^{-j2\mathrm{k}_0 \cdot \Delta \mathrm{r}_p} \right|^2 \approx N$$

The pathological case were no scatter is received due to perfect destructive interference will almost surely never happen with a large number of electrons.

Debye Length

At what scale are collective effects important?

- Characteristic particle velocity: Electron thermal speed

$$v_{te} = \sqrt{\frac{k_B T_e}{m_e}}$$

- Characteristic time scale for collective interacttions: Inverse electron plasma frequency

$$\tau_e = \frac{1}{\omega_{pe}} = \sqrt{\frac{m_e \epsilon_0}{e^2 N_e}}$$

- Characteristic length scale: Debye length

$$\lambda_{De} = v_{te}\tau_e = \sqrt{\frac{\epsilon_0 k_B T_e}{e^2 N_e}}$$

Collective effects will have a significant affect on the particle trajectories over a Bragg wavelength if $\lambda_b > \lambda_{De}$

Collective vs Non-Collective Regimes

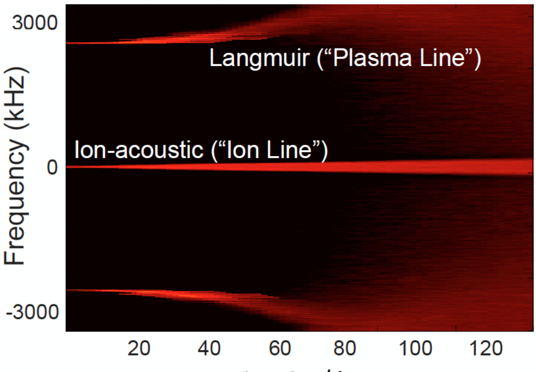
- Non-Collective Regime: $k_b^2 \lambda_{De}^2 \gg 1$
  - Electron trajectories are effectively independent
  - Doppler spectrum is the velocity spectrum of the electrons (e.g. a Maxwellian)
  - Spectral width is determined by electron thermal velocity (very wide)
- Collective Regime: $k_b^2 \lambda_{De}^2 \ll 1$
  - Electron trajectories are coupled to each other, and to the ions
  - Scatter is Bragg scatter from waves in the plasma whose wavelengths match the Bragg wavelength
  - Doppler spectrum has peaks at the phase-velocities of those plasma waves
  - Spectral width is determined by ion acoustic speed (narrow)

Spectrum of Density Fluctuations

- Thermal plasmas are filled with ambient density fluctuations.
- The spectra of the ambient fluctuations peak around ω, k pairs that satisfy a dispersion relation for a plasma normal mode.
- An ISR would pick out one slice of this spectrum at $\mathrm{k} = \mathrm{k}_b$.

Computed density fluctuation spectrum from PIC simulations (Diaz et al., RS [2008])

$$\left\langle \left| n_e \left(\mathrm{k}, \omega\right) \right|^2 \right\rangle$$



Filtered Noise

Can you hear the ocean in a seashell?

Resonant Filter

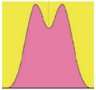
Ambient Noise → → Colored Noise (Ocean Sounds)

Analogy to ISR:

Thermal Motion → Collective Effects → ISR Spectrum

$\left\langle \left| n_{ti}\left(\mathbf{k},\omega\right)\right|^2 \right\rangle$

$\frac{|\chi_e|^2}{|1+\chi_i+\chi_e|^2}$



Bragg Scatter Summary

- Scatter from targets spaced by the Bragg wavelength ($\lambda/2$) add constructively
- Scatter from a large number of electrons samples the Fourier transform of the electron density distribution at the Bragg wavenumber
- Thermal plasmas are naturally full of a whole spectrum of waves
- ISR is Bragg scatter from those thermal waves that match the Bragg wavenumber