Phasor Analysis for Phased Arrays

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Antenna Fundamentals

For an antenna carrying current $i(t)$, the far field vector potential and electric fields will be of the form

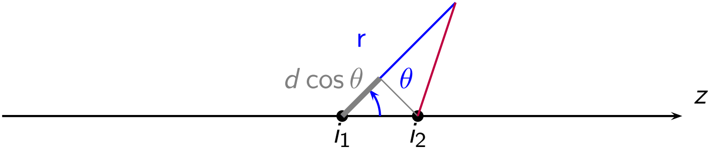
$$\mathrm{A}(\mathrm{r}, t) \propto \frac{1}{r} i\left(t - r/c\right)$$
$$\mathrm{E}(\mathrm{r}, t) \propto -\frac{\partial \mathrm{A}}{\partial t}$$

Note in the special case of plane waves in the Lorenz Gauge
$\mathrm{E} = -\nabla\phi - \frac{\partial \mathrm{A}}{\partial t} = -\frac{\partial}{\partial t}\left(\hat{k} \times \mathrm{A} \times \hat{k}\right)$.
Instantaneous Power

$$S(\mathrm{r}, t) = \frac{1}{\mu_0} \mathrm{E} \times \mathrm{B} = \frac{1}{\eta_0} |\mathrm{E}|^2$$

Note $\eta_0 = \sqrt{\mu_0/\epsilon_0} \approx 377\ \Omega$

Simple Two Element Array Example



Suppose $d = \lambda/4$ and

$$i_1(t) = \cos(\omega t)$$
$$i_2(t) = \cos\left(\omega t + \frac{\pi}{2}\right)$$

How does the radiated power vary as a function of θ?

Time-Domain Solution

$$
\begin{aligned}
\mathrm{A}(\mathrm{r}, t) &\propto \frac{1}{r} i_1 (t - r/c) + \frac{1}{r} i_2 (t - (r - d\cos\theta)/c) \\
&\propto \frac{1}{r}\cos[\omega(t - r/c)] + \frac{1}{r}\cos\left[\omega\left(t - (r - d\cos\theta)/c + \frac{\pi}{2}\right)\right] \\
&\propto \frac{1}{r}\cos[\omega t - kr] + \frac{1}{r}\cos\left[\omega t - kr + kd\cos\theta + \frac{\pi}{2}\right] \\
\mathrm{E}(\mathrm{r}, t) &\propto -\frac{\partial \mathrm{A}}{\partial t} \\
&\propto \frac{\omega}{r}\sin[\omega t - kr] + \frac{\omega}{r}\sin\left[\omega t - kr + kd\cos\theta + \frac{\pi}{2}\right] \\
S &\propto |\mathrm{E}|^2
\end{aligned}
$$

Trigonometric Identity

$$A\sin\left(x+\theta_a\right)+B\sin\left(x+\theta_b\right)=K\sin\left(x+\phi\right)$$

where

$$K^2=A^2+B^2+2AB\cos\left(\theta_b-\theta_a\right)$$
$$\tan\phi=\frac{A\sin\theta_a+B\sin\theta_b}{A\cos\theta_a+B\cos\theta_b}$$

Therefore

$$\mathrm{E}\left(\mathrm{r},t\right)\propto\frac{\omega}{r}K\sin\left[\omega t-kr+\phi\right]$$
$$\left|\mathrm{E}\right|^2\propto\frac{\omega^2}{r^2}K^2\sin^2\left[\omega t-kr+\phi\right]$$
$$K^2=2+2\cos\left(kd\cos\theta+\frac{\pi}{2}\right)$$

Average Power

Average power over a full wave period

$$
\begin{aligned}
\langle S \rangle &= \frac{\omega}{2\pi} \int_0^{2\pi/\omega} S(t) dt \\
&\propto \frac{\omega}{2\pi} \int_0^{2\pi/\omega} |\mathrm{E}|^2 \, dt \\
&\propto \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \frac{\omega^2}{r^2} K^2 \sin^2 \left[\omega t - kr + \phi \right] dt
\end{aligned}
$$

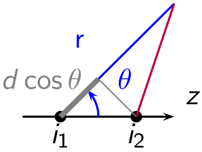
Integral identity

$$
\frac{\omega}{2\pi} \int_0^{2\pi/\omega} \sin^2 \left[\omega t + \theta \right] dt = \frac{1}{2}
$$

Therefore

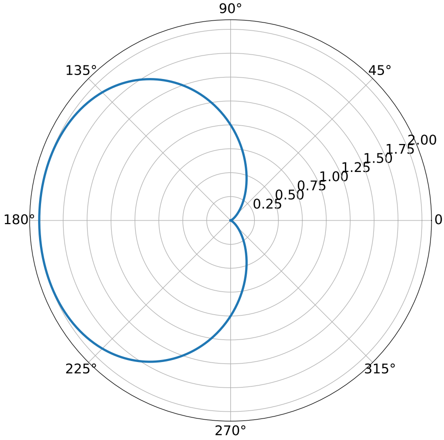
$$
\langle S \rangle \propto \frac{\omega^2}{r^2} \frac{K^2}{2}
$$

Radiation Pattern



$$\langle S \rangle \propto \frac{K^2}{2} = 1 + \cos\left(\frac{\pi}{2}\cos\theta + \frac{\pi}{2}\right)$$

For $d = \lambda/4$, $kd = \pi/2$



Easier Way: Using Phasors

$$K\cos(\omega t+\phi) = \Re\left\{Ke^{j\phi}e^{j\omega t}\right\}$$
$$= \Re\left\{\tilde{K}e^{j\omega t}\right\}$$

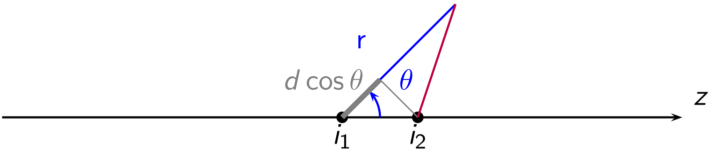
Antenna basics simplify considerably

$$i(t) \Rightarrow \tilde{I}$$
$$\mathrm{A}(\mathrm{r},t) \propto \frac{1}{r} i\left(t - r/c\right) \Rightarrow \tilde{\mathrm{A}} \propto \frac{1}{r}\tilde{I}e^{-jkr}$$
$$\mathrm{E}(\mathrm{r},t) \propto -\frac{\partial \mathrm{A}}{\partial t} \Rightarrow \tilde{\mathrm{E}} \propto -j\omega\tilde{A} \propto \frac{-j\omega}{r}\tilde{I}e^{-jkr}$$

Average Power

$$\langle S\rangle \propto \frac{1}{2}\left|\tilde{E}\right|^2$$

Two Element Array Example Again With Phasors



$$\tilde{I}_1 = 1 \qquad \tilde{I}_2 = e^{j\pi/2}$$

$$\begin{aligned}
\tilde{E} &= \frac{-j\omega}{r}\tilde{I}_1 e^{-jkr} + \frac{-j\omega}{r}\tilde{I}_2 e^{-jkr+jkd\cos\theta} \\
&= \frac{-j\omega}{r} e^{-jkr} + \frac{-j\omega}{r} e^{-jkr+jkd\cos\theta+j\pi/2} \\
&= \frac{-j\omega}{r} e^{-jkr}\left[1 + e^{jkd\cos\theta+j\pi/2}\right]
\end{aligned}$$

$$\langle S\rangle \propto \frac{\omega^2}{r^2}\frac{1}{2}\left|1 + e^{jkd\cos\theta+j\pi/2}\right|^2$$

Equivalence of the Solutions

$$
\begin{aligned}
\langle S \rangle &\propto \frac{1}{2}\left|1 + e^{j\frac{\pi}{2}\cos\theta + j\frac{\pi}{2}}\right|^2 \\
&= \frac{1}{2}\left|e^{j\frac{\pi}{4}\cos\theta + j\frac{\pi}{4}}\left(e^{-j\frac{\pi}{4}\cos\theta - j\frac{\pi}{4}} + e^{j\frac{\pi}{4}\cos\theta + j\frac{\pi}{4}}\right)\right|^2 \\
&= \frac{1}{2}\left|e^{j\frac{\pi}{4}\cos\theta + j\frac{\pi}{4}} 2\cos\left(\frac{\pi}{4}\cos\theta + \frac{\pi}{4}\right)\right|^2 \\
&= 2\cos^2\left(\frac{\pi}{4}\cos\theta + \frac{\pi}{4}\right) \\
&= 1 + \cos\left(\frac{\pi}{2}\cos\theta + \frac{\pi}{2}\right)
\end{aligned}
$$