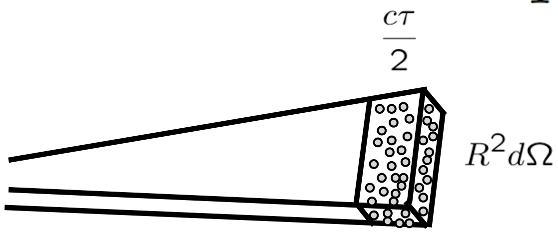
ISR Theory: The SHORT Summary

P. J. Erickson ISR School 2023

Remote sensing a plasma: The experimental (radar) view

Suppose we transmit a wave towards a plasma and measure the scattered wave:

$$P_{rec} = (P_{inc}) A_{scat} \left(\frac{A_{rec}}{4\pi R^2}\right)$$



$$A_{scat} = \sigma_{radar} V_s$$

(ionosphere is a beam filling target)

$$\sigma_{radar} = 4\pi \sigma_{total}$$

(Solid angle)

$$\left(\frac{P_{rec}}{P_{inc}}\right)\left(\frac{4\pi R^2}{A_{rec}}\right)\left(\frac{1}{V_s}\right) = 4\pi r_e^2 \sin^2\delta \left\langle |\Delta N(k)|^2 \right\rangle$$

Measurable experimentally

Physics info is here!

(1 for backscatter)

Radar cross-section of ionospheric plasma

Assume a beam filling plasma at F region altitudes (300 km) with very high electron density (1E12 electrons per m3 - BEST CASE):

Classical electron scattering cross-section $\sigma_e = 10^{-28} m^2/e^-$

Assume a pulse length of 10 km.
Assume a cross-beam width of 1 km (~ Arecibo).

Total cross section is then (10 km x 1 km x 1 km x 1E12 m^-3 x 1E-28 m^-2/e-):

$$\boxed{\sigma_{tot} \sim 10^{-6} m^2}$$

-60 dBsm! Are we going to be able to do this at all?

NB: Born approximation is very valid, since total amount of scattered power in the volume ~ 1E-12. So we can make full range profiles if we can detect the scatter.

Detectability of scatter from ionospheric plasma

For fraction of scattered power actually received, assume isotropic scatter and a BIG ~100 m diameter antenna:

$$f_{rec} = \frac{A_{rec}}{4\pi R^2} \sim \frac{10^4\ m^2}{4(300 \times 10^3\ m)^2}$$

About -80 dB (1E-8): not much. So:

$$\frac{P_{rec}}{P_{tx}} \sim 10^{-20}$$

So a radar with 1 MW transmitted signal receives 10 femtowatts of incoherently scattered power from free electrons in the ionosphere.

REALLY not very much.

Detectability of scatter from ionospheric plasma

What matters, though, is the signal to noise ratio:

$$P_{noise} = (k_B T_{eff})\,(BW)$$ (derived later)

Typical effective noise temperatures ~100 to 200 K at UHF frequencies (430 MHz, say).

Assume the bandwidth is set by thermal electron motions in a Boltzmann sense:

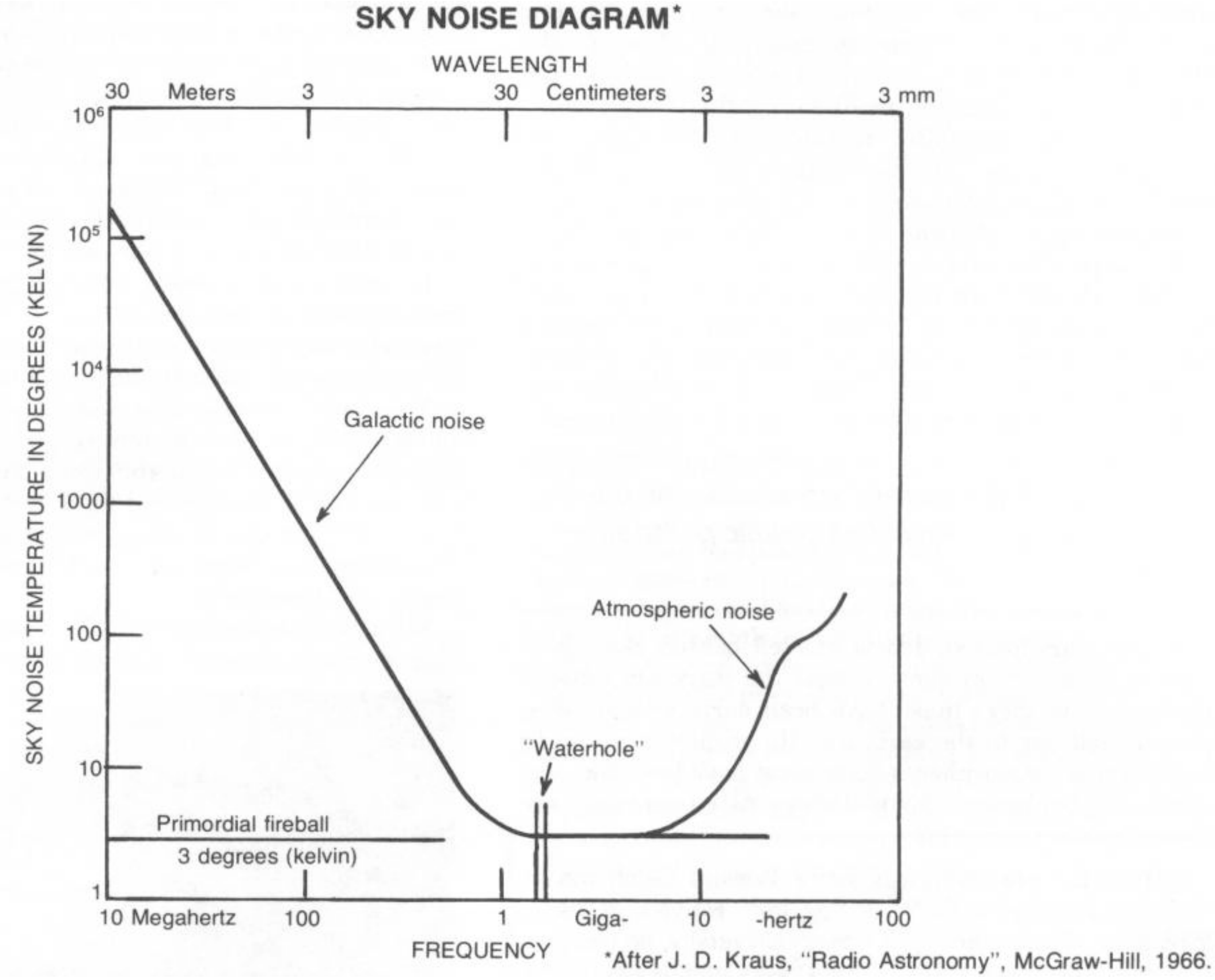
$$3k_B T_e \sim m_e v_{e,th}^2$$

$$v_{e,th} \sim \sqrt{\frac{3k_B T_e}{m_e}} \sim 2 \times 10^5 m/s$$

$$BW \sim (v_{e,th})\,(2)(2)(\frac{f_{tx}}{c}) \sim 10^6 Hz$$

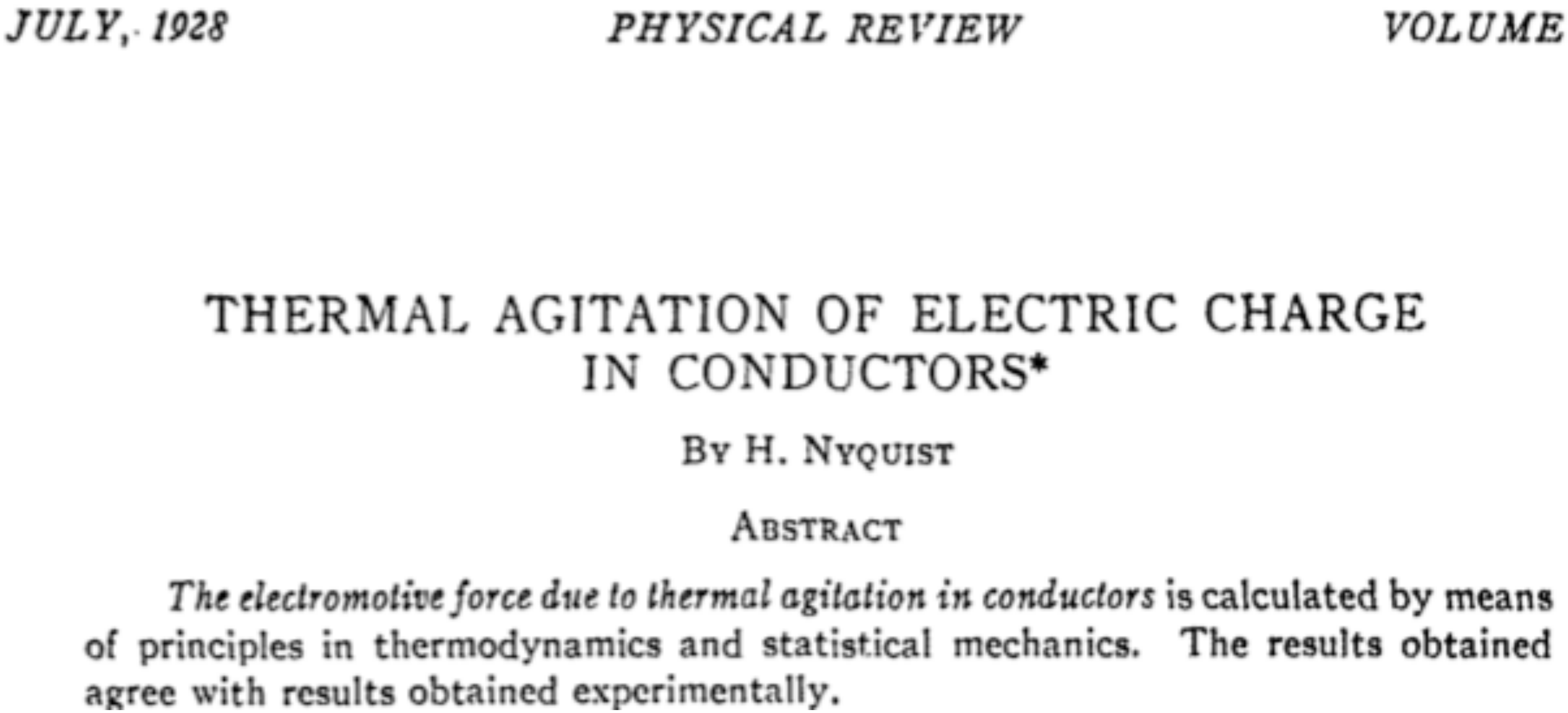
(2s are for up/down, backscatter)

Sky Noise: The Universe Is Also Transmitting



*After J. D. Kraus, "Radio Astronomy", McGraw-Hill, 1966.

Nyquist-Johnson thermal noise (1928): Always present



JULY, 1928 *PHYSICAL REVIEW* *VOLUME 32*

THERMAL AGITATION OF ELECTRIC CHARGE IN CONDUCTORS*

BY H. NYQUIST

ABSTRACT

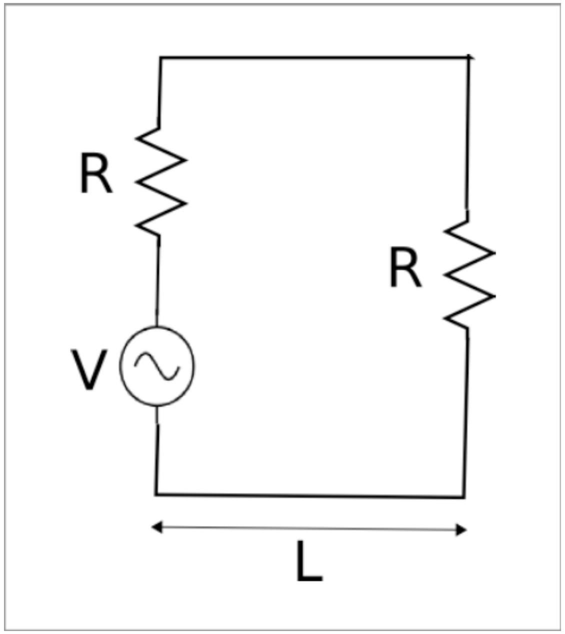
The electromotive force due to thermal agitation in conductors is calculated by means of principles in thermodynamics and statistical mechanics. The results obtained agree with results obtained experimentally.

H. Nyquist 1889-1976
(born Nilsby, Sweden)

"Bert" Johnson 1887-1970
(born Gothenburg, Sweden)

Nyquist-Johnson thermal noise (1928)

Circuit loop has a temperature T
(with thermal vibrations)



(Each resistor)

$$P_{absorbed} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \; \frac{h|\omega|}{e^{\frac{h|\omega|}{k_B T}} - 1}$$

$$I = \frac{V}{2R}$$

$$P_{emitted} = \frac{\langle V^2 \rangle}{4R} \longleftarrow S(\omega)$$

(Each resistor)

Voltage spectral density in the circuit loop (= per frequency):

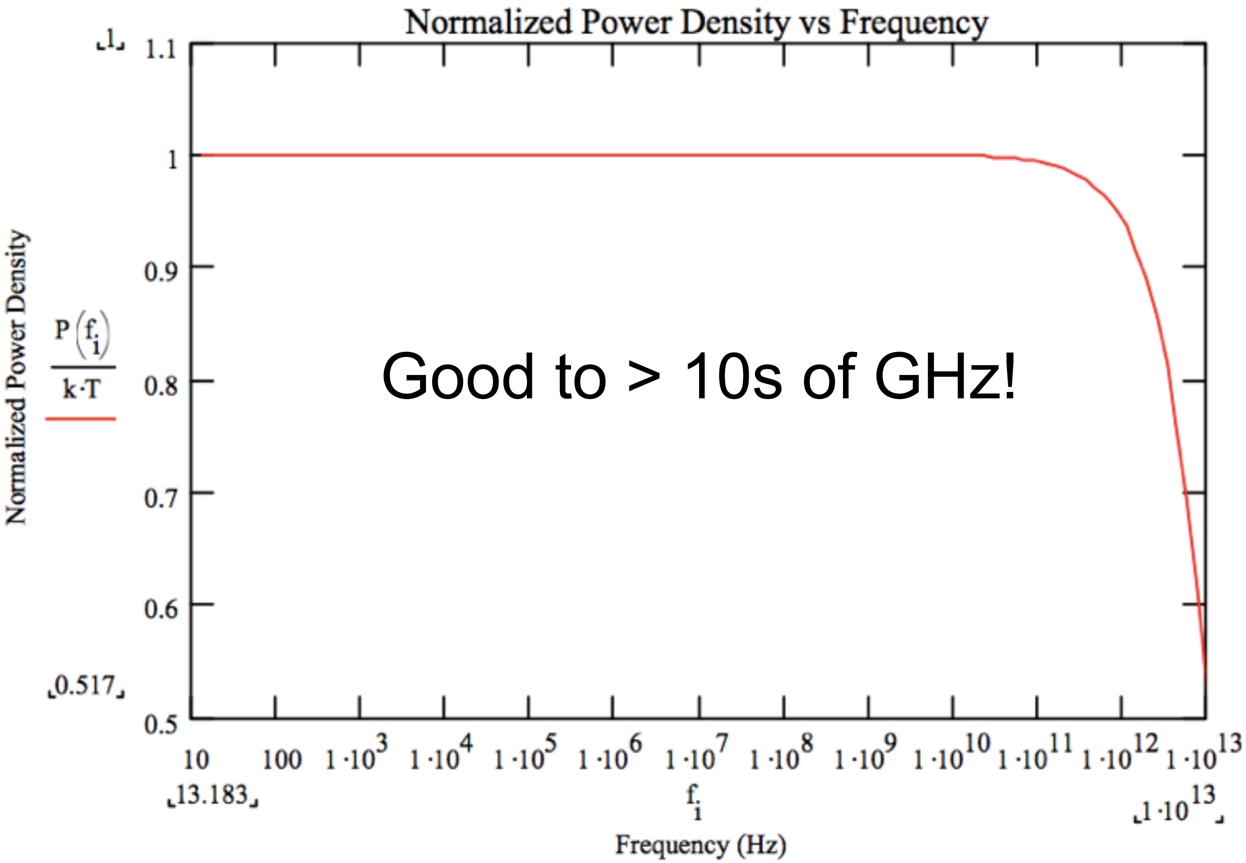
$$S(\omega) = 4R \frac{h|\omega|}{e^{\frac{h|\omega|}{k_B T}} - 1}$$

$$h\omega \ll k_B T : S(\omega) = 4R k_B T$$

Over a range of frequencies:

$$\boxed{P(\Delta f) \propto k_B T \, \Delta f}$$

Nyquist-Johnson thermal noise (1928)



Good to > 10s of GHz!

(Clay Turner, Wireless Solutions, 2007)

Detectability of scatter from ionospheric plasma

Finally,

$$P_{noise} \sim 2 \times 10^{-15} W$$

$$S/N \sim 5$$

Workable!

But you need a megawatt class transmitter and a huge antenna.

1950s: technology makes this possible (radio astronomy + construction = large antennas, military needs = high power transmitters)

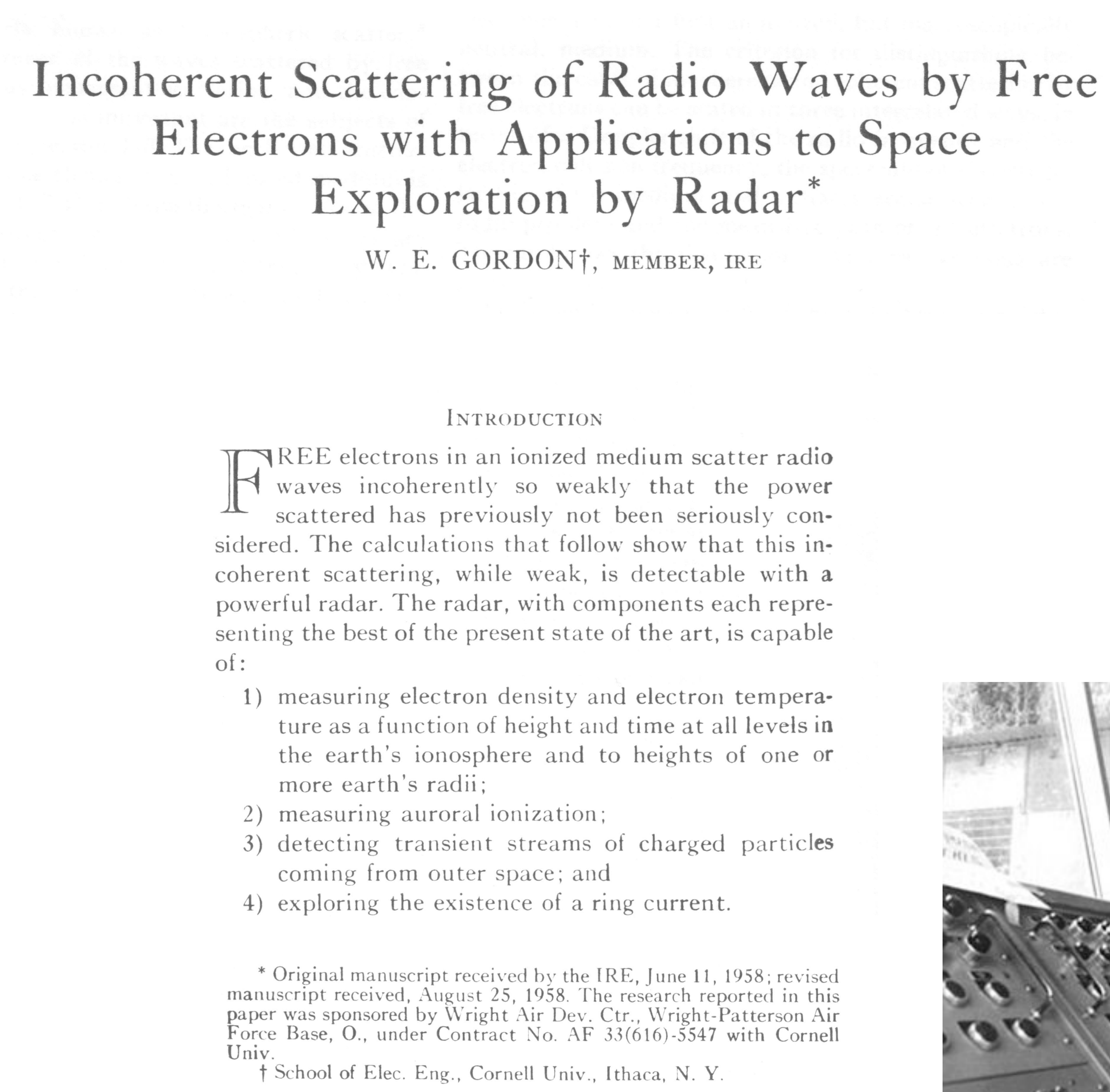
Early Incoherent Scatter Radar

- W. E. Gordon of Cornell is credited with the idea for ISR.
- *"Gordon (1958) has recently pointed out that scattering of radio waves from an ionized gas in thermal equilibrium may be detected by a powerful radar."* (Fejer, 1960)
- Gordon proposed the construction of the Arecibo Ionospheric Observatory for this purpose (NOT for radio astronomy as the primary application)
- Many different derivations in late 1950s/early 1960s

~40 megawatt-acres

- 1000' Diameter Spherical Reflector
  - 62 dB Gain
- 430 MHz line feed 500' above dish
- Gregorian feed
- Steerable by moving feed.

Proceedings of the IRE, November 1958



Incoherent Scattering of Radio Waves by Free Electrons with Applications to Space Exploration by Radar[*]

W. E. GORDON[†], MEMBER, IRE

INTRODUCTION

FREE electrons in an ionized medium scatter radio waves incoherently so weakly that the power scattered has previously not been seriously considered. The calculations that follow show that this incoherent scattering, while weak, is detectable with a powerful radar. The radar, with components each representing the best of the present state of the art, is capable of:

1) measuring electron density and electron temperature as a function of height and time at all levels in the earth's ionosphere and to heights of one or more earth's radii;
2) measuring auroral ionization;
3) detecting transient streams of charged particles coming from outer space; and
4) exploring the existence of a ring current.

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[*] Original manuscript received by the IRE, June 11, 1958; revised manuscript received, August 25, 1958. The research reported in this paper was sponsored by Wright Air Dev. Ctr., Wright-Patterson Air Force Base, O., under Contract No. AF 33(616)-5547 with Cornell Univ.

[†] School of Elec. Eng., Cornell Univ., Ithaca, N. Y.

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Early Experiments in Incoherent-Scatter Radar

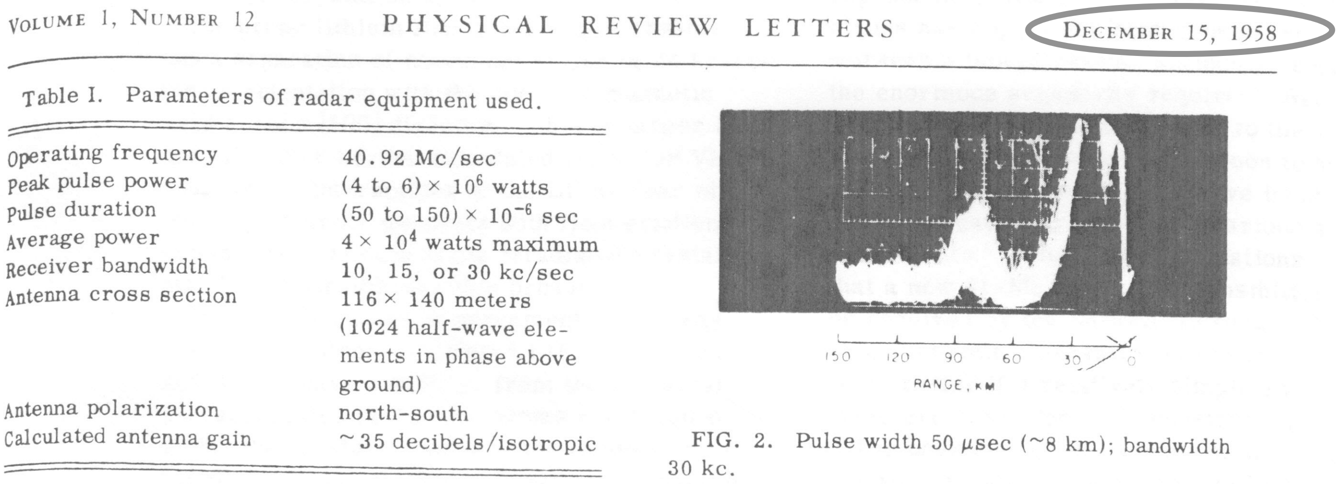
- **K.L. Bowles [Cornell PhD 1955]**, Observations of vertical incidence scatter from the ionosphere at 41 Mc/sec. *Physical Review Letters* 1958:

"*The possibility that incoherent scattering from electrons in the ionosphere, vibrating independently, might be observed by radar techniques has apparently been considered by many workers although seldom seriously because of the enormous sensitivity required…*"

First Incoherent-Scatter Radar

…*Gordon (*W.E. Gordon from Cornell*) recalled this possibility to the writer* [spring 1958; D. T. Farley] *while remarking that he hoped soon to have a radar sensitive enough to observe electron scatter in addition to various astronomical objects…”*

Bowles executed the idea - hooked up a large transmitter to a dipole antenna array in Long Branch Ill., took a few measurements.



VOLUME 1, NUMBER 12 PHYSICAL REVIEW LETTERS DECEMBER 15, 1958

Table I. Parameters of radar equipment used.

Operating frequency	40.92 Mc/sec
Peak pulse power	(4 to 6) × 10^6 watts
Pulse duration	(50 to 150) × 10^{-6} sec
Average power	4 × 10^4 watts maximum
Receiver bandwidth	10, 15, or 30 kc/sec
Antenna cross section	116 × 140 meters (1024 half-wave elements in phase above ground)
Antenna polarization	north-south
Calculated antenna gain	~35 decibels/isotropic

FIG. 2. Pulse width 50 μsec (~8 km); bandwidth 30 kc.

~6 week setup time

Oscilloscope + camera + ~4 sec exposure
(10 dB integration)

Incoherent Scattering Detectability

Bowles' results found approximately the expected amount of power scattered from the electrons (scattering is proportional to charge to mass ratio - electrons scatter the energy).

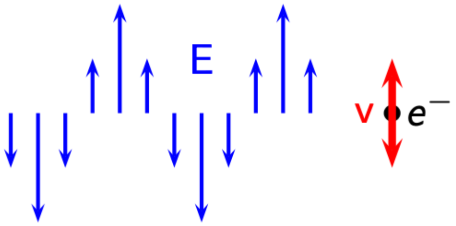
BUT: his detection with a 20 megawatt-acre system at 41 MHz (***high*** cosmic noise background; lots of noise) implies a spectral width **100x** narrower than expected – almost as if the much heavier (and slower) ions were controlling the scattering spectral width.

In fact, they do.

Thomson Scatter Summary

(Collective Thomson scatter == incoherent scatter)

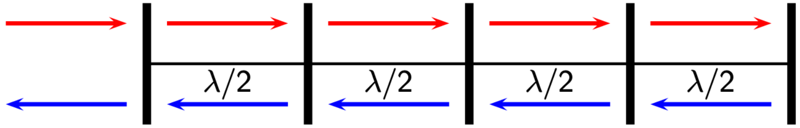
- Thomson scatter from electrons is a fundamental physical process
- Radar cross section of one electron is a constant independent of wavelength ($\sim 10^{-28}$ m^2)
- Scatter from ions is negligible
- Even though one electron has a tiny cross section, scatter can still be detectable from a whole volume of electrons



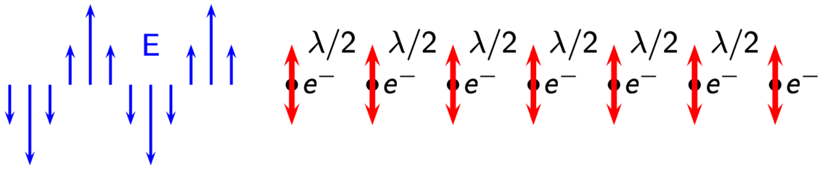
Bragg Scatter Summary

- Scatter from targets spaced by the Bragg wavelength ($\lambda/2$) add constructively
- Scatter from a large number of electrons samples the Fourier transform of the electron density distribution at the Bragg wavenumber
- Thermal plasmas are naturally full of a whole spectrum of waves
- ISR is Bragg scatter from those thermal waves that match the Bragg wavenumber

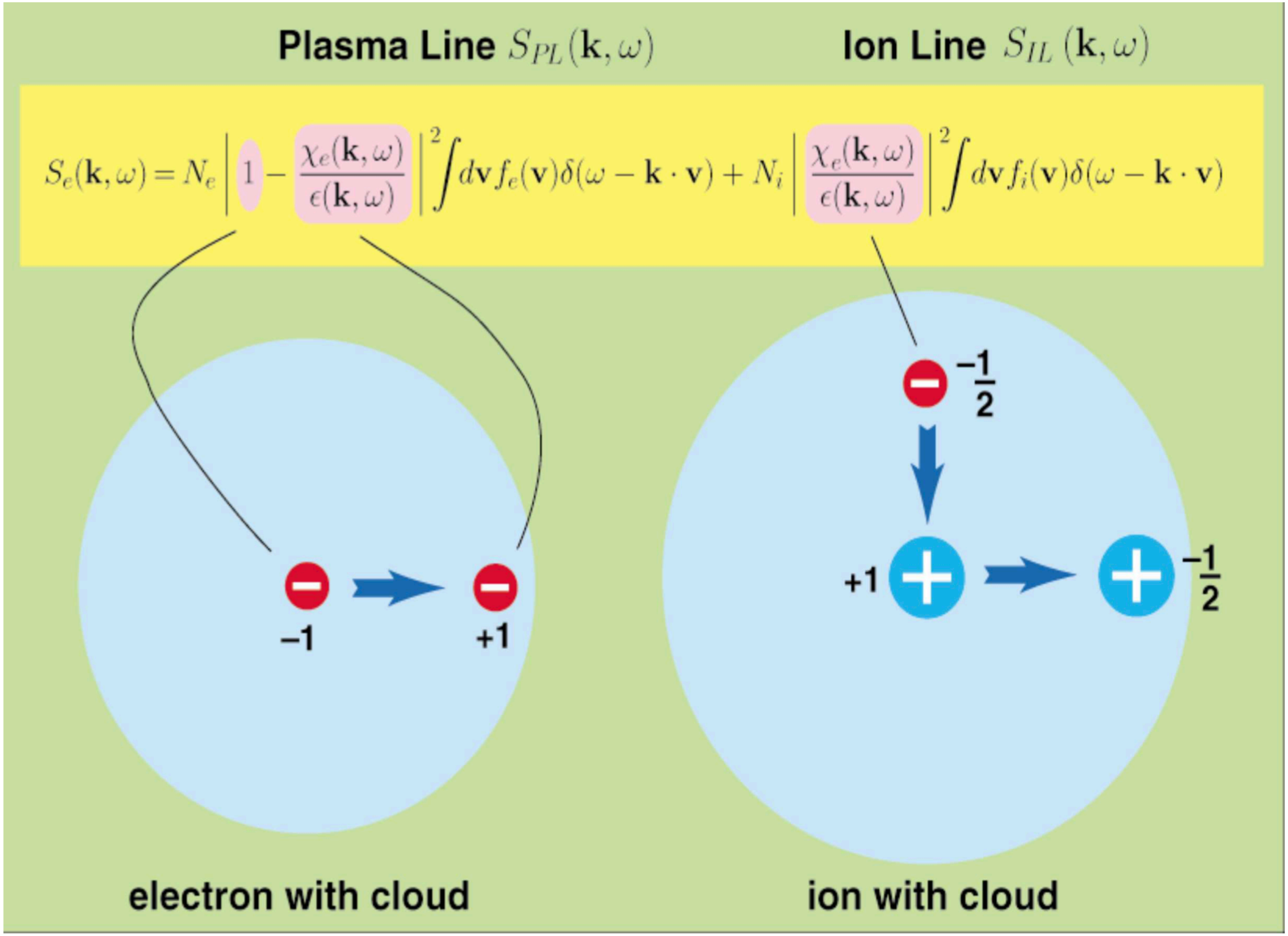
Stack of reflecting structures



Stack of electrons



Dressed Particle Concepts



Plasma Line $S_{PL}(\mathbf{k},\omega)$ **Ion Line** $S_{IL}(\mathbf{k},\omega)$

$$S_e(\mathbf{k},\omega) = N_e \left| 1 - \frac{\chi_e(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)} \right|^2 \int d\mathbf{v}\, f_e(\mathbf{v})\delta(\omega - \mathbf{k}\cdot\mathbf{v}) + N_i \left| \frac{\chi_e(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)} \right|^2 \int d\mathbf{v}\, f_i(\mathbf{v})\delta(\omega - \mathbf{k}\cdot\mathbf{v})$$

Total Scattered Power Summary

- In the collective regime $\sigma \neq \sigma_e V N_e$
- Correction terms can be understood using dressed particle theory concepts
- Corrections depend on temperature ratio (T_e/T_i) and Debye length
- ISR typically report both uncorrected N_e (from power) and corrected N_e (from fitted ACFs)
- Dressed particle theory concepts also explain enhanced plasma line observations

- **Non-Collective Limit:** $k^2\lambda_{De}^2 \gg 1$
  Electron line dominates (wide bandwidth)

$$\sigma = \sigma_e V N_e$$

- **Collective Limit:** $k^2\lambda_{De}^2 \ll 1$
  Ion line dominates (narrow bandwidth)

$$\sigma = \sigma_e V \frac{N_e}{1+\frac{T_e}{T_i}}$$

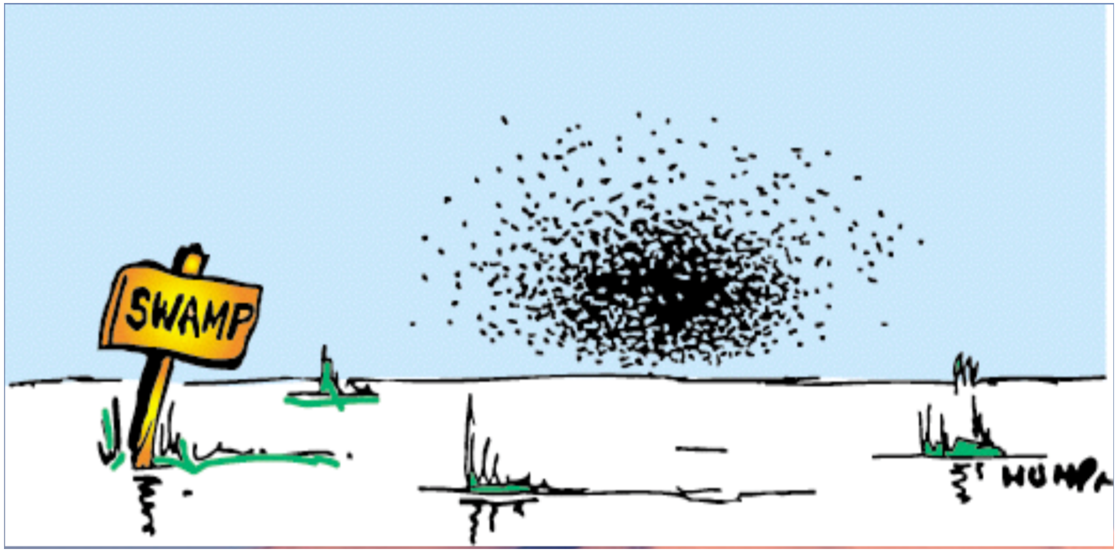
Reporting Electron Density from Ion Line Power

Ion Line Cross Section:

$$\boxed{\sigma = \sigma_e V \frac{N_e}{2} \zeta}$$

Temperature Correction: $\boxed{\zeta = \frac{2}{\left(1 + k^2 \lambda_{De}^2\right)\left(1 + k^2 \lambda_{De}^2 + T_e/T_i\right)}}$

- Uncorrected N_e: Assume $\zeta = 1$.
    - $T_e/T_i = 1$
    - $k^2 \lambda_{De}^2 \ll 1$.
- N_e with model: Compute ζ using an empirical model of T_e/T_i as a function of altitude.
- N_e with fits: Compute ζ with T_e and T_i estimated from fitted ACF.



Radio Noise

Nyquist Noise Theorem: $\boxed{P_N = k_B T_{sys} B}$

- A good UHF receiver will have a $T_{sys} \approx 125\ \mathrm{K}$.
- B is the receiver bandwidth.

Doppler shift from electron thermal motion:

$$\Delta f = \frac{2}{c} f_{\mathrm{Tx}} v \approx \frac{2}{c} f_{\mathrm{Tx}} \sqrt{\frac{k_B T_e}{m_e}}$$

Let's assume we need to capture $B = 4\Delta f$ to get the full spectrum.

For $f_{\mathrm{Tx}} = 450\ \mathrm{MHz}$ and $T_e = 1000\ \mathrm{K}$:

$$B = 1.48\ \mathrm{MHz} \Rightarrow \boxed{P_N = 2.55 \times 10^{-15}\ \mathrm{W}}$$

Electrons control bandwidth
(no collective effects)
Need a >100m class
antenna (Arecibo)!

What if instead the bandwidth is related to the ion motion?

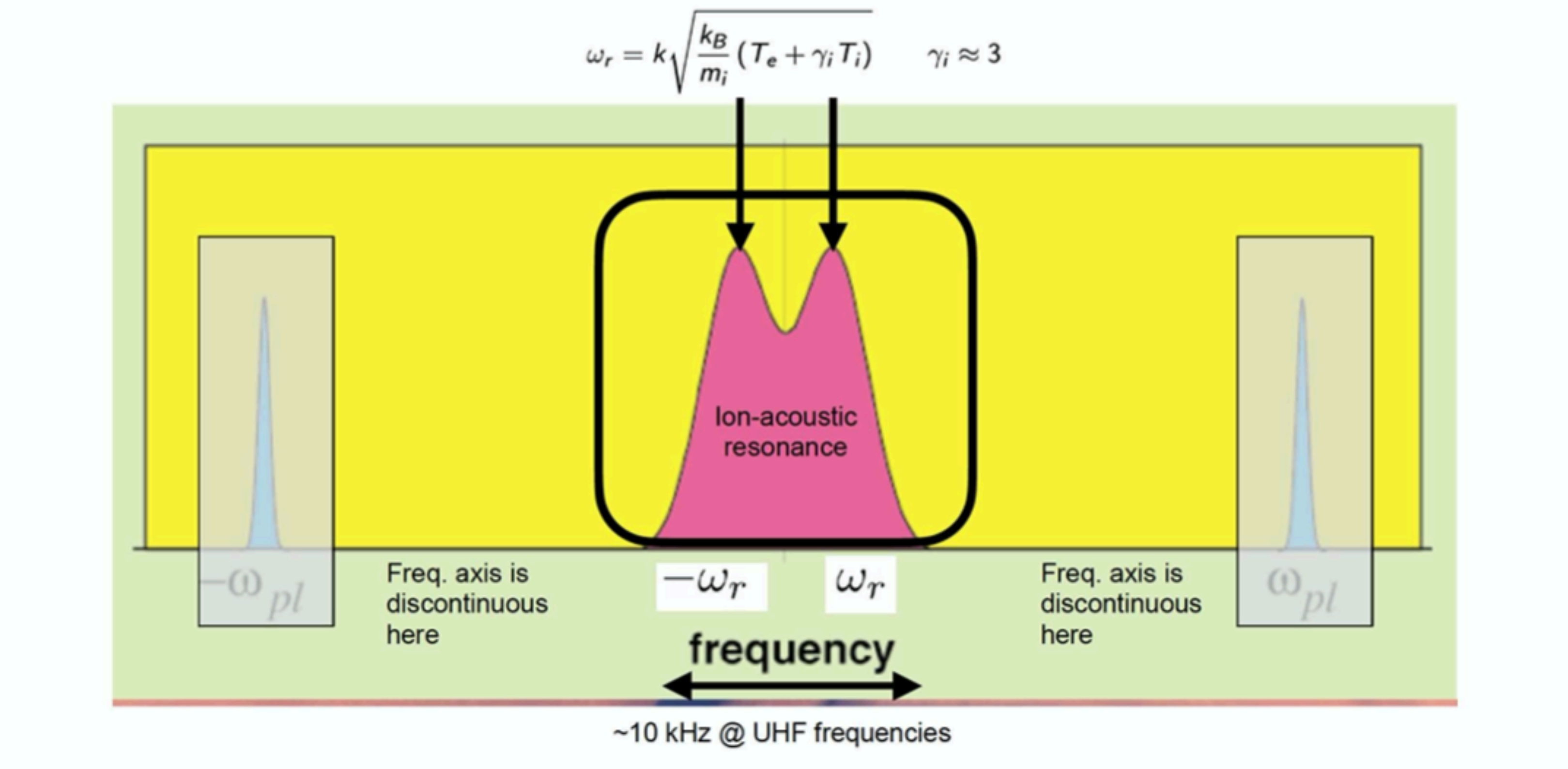
$$v_i = \sqrt{\frac{m_e}{m_i}} v_e \Rightarrow v_i = 5.83 \times 10^{-3} v_e \text{ for } \mathrm{O}^+$$

The same numbers would yield

$$B = 8.63\mathrm{kHz} \Rightarrow \boxed{P_N = 1.48 \times 10^{-17}\ \mathrm{W}}$$

Ions control bandwidth
(collective effects)
Much smaller bandwidth

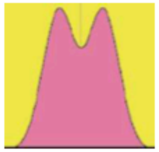
IS Spectrum: Ion Line



$$\omega_r = k\sqrt{\frac{k_B}{m_i}(T_e+\gamma_i T_i)} \qquad \gamma_i \approx 3$$

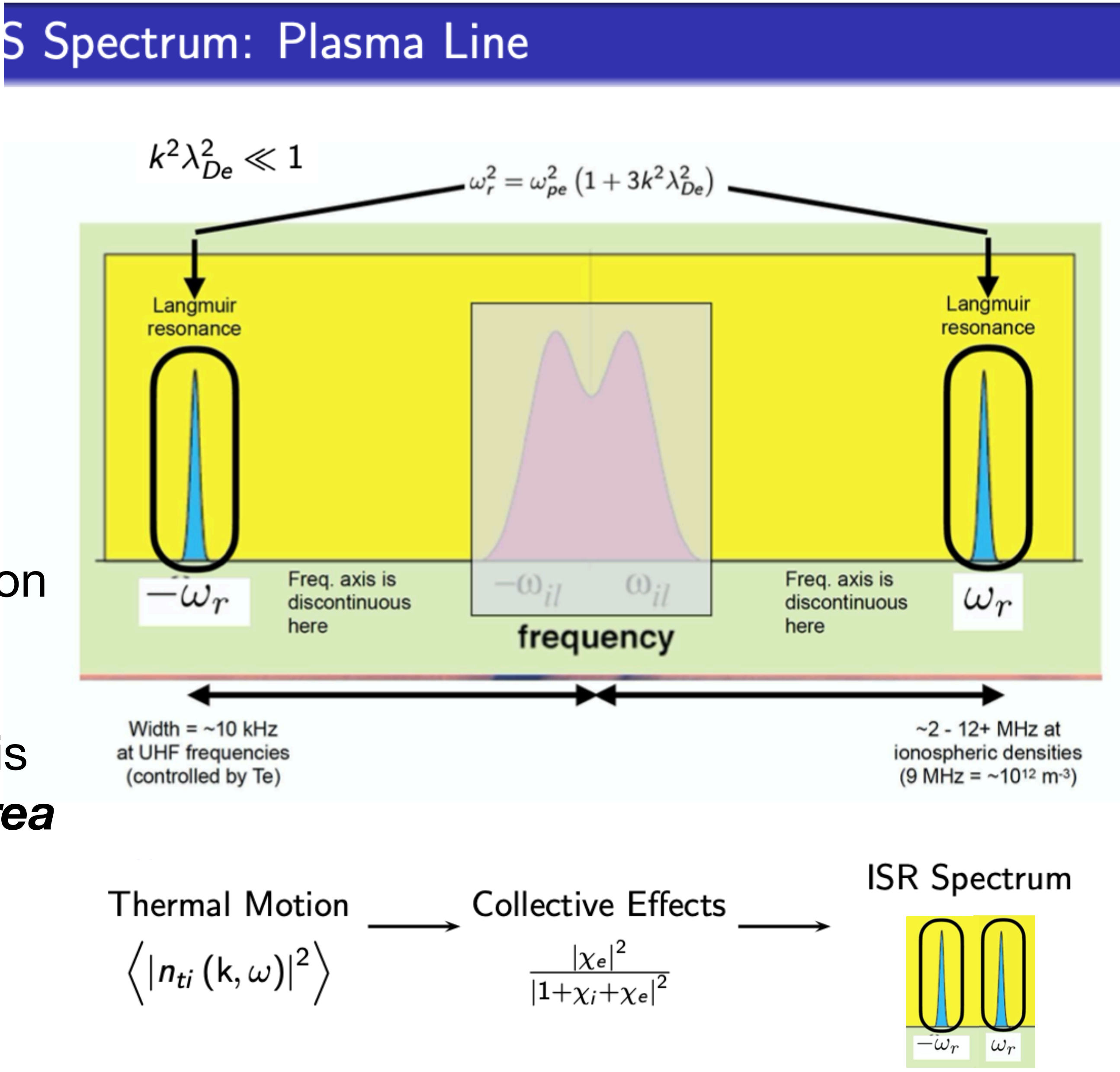
Thermal Motion $\longrightarrow$ Collective Effects $\longrightarrow$ ISR Spectrum

$$\left\langle |n_{ti}(k,\omega)|^2 \right\rangle \longrightarrow \frac{|\chi_e|^2}{|1+\chi_i+\chi_e|^2} \longrightarrow$$



$$\boxed{k^2\lambda_{De}^2 \ll 1}$$

- Weak!
- Typically daytime only (enhanced photoelectron fluxes)
- Precise when visible: primary measurement is a ***frequency***, not an ***area***



S Spectrum: Plasma Line

$k^2\lambda_{De}^2 \ll 1$

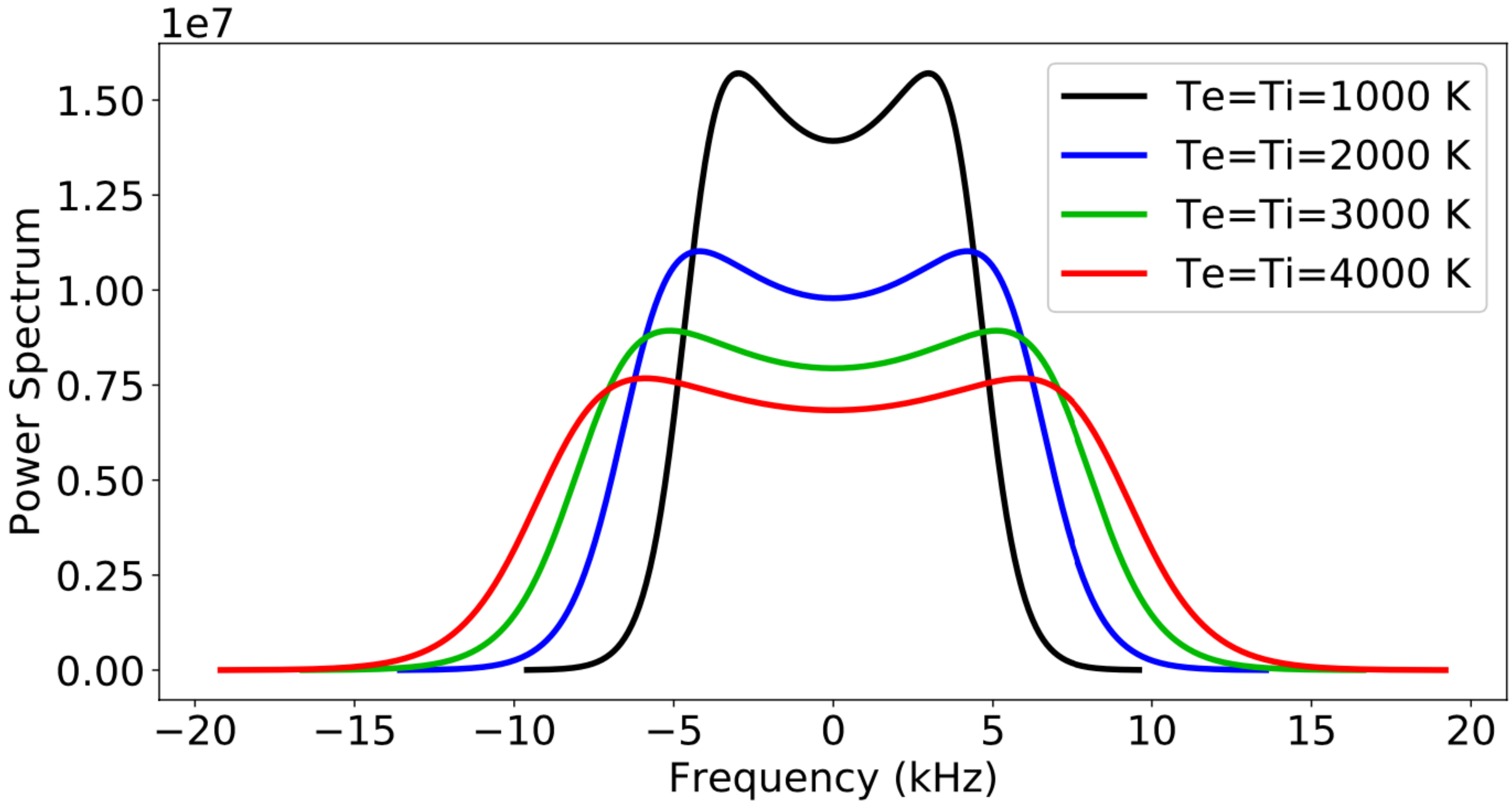
$\omega_r^2 = \omega_{pe}^2\left(1 + 3k^2\lambda_{De}^2\right)$

$$\text{Thermal Motion} \longrightarrow \text{Collective Effects} \longrightarrow \text{ISR Spectrum}$$

$$\left\langle \left| n_{ti}(k,\omega)\right|^2 \right\rangle \qquad \frac{|\chi_e|^2}{|1+\chi_i+\chi_e|^2}$$

Ion-Acoustic Line ("Ion Line")

Temperature Effects ($T_e/T_i = 1$)



$f = 449.3$ MHz $\quad N_e = 3 \times 10^{11}$ m^{-3} $\quad m_i = 16$ amu