Introduction to Radar Statistics

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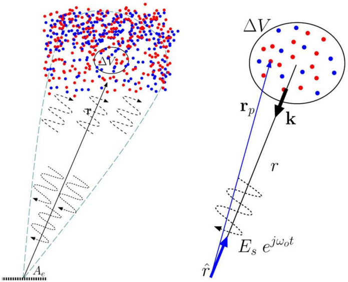
The Need for Statistical Descriptions of ISR Signals

If I knew the positions of every single electron in the scattering volume, I would know the received voltage exactly:



Exact expression for scattered electric field as a superposition of Thomson scatterers:

$$E_s = -\frac{r_e}{r} E_0 \sum_{p=1}^{N_0 \Delta V} e^{j\mathbf{k}\cdot\mathbf{r}_p}$$

ISR theory predicts statistical aspects of the scattered signal:

Scattered Power: $\left\langle |E_s|^2 \right\rangle$ Autocorrelation Function: $\langle E_s(t) E_s^*(t-\tau) \rangle$

These statistical properties are functions of macroscopic properties of the plasma: N_e, T_e, T_i, u_{los}.

Statistical Properties of ISR Voltages

ISR signals are complex-valued, zero-mean, random phase, Gaussian random variables with variances related to their power P:

$$
\begin{aligned}
V &= V_R + jV_I \\
E\left\{V_R\right\} &= E\left\{V_I\right\} = 0 \\
E\left\{V_R^2\right\} &= E\left\{V_I^2\right\} = \frac{1}{2}P \qquad E\left\{V_R V_I\right\} = 0 \\
E\left\{|V|^2\right\} &= E\left\{V_R^2 + V_I^2\right\} = P \\
E\left\{V_R^4\right\} &= E\left\{V_I^4\right\} = \frac{3}{4}P^2 \qquad E\left\{V_R^2 V_I^2\right\} = E\left\{V_R^2\right\}E\left\{V_I^2\right\} = \frac{1}{4}P^2 \\
Var\left\{|V|^2\right\} &= E\left\{\left(|V|^2\right)^2\right\} - \left(E\left\{|V|^2\right\}\right)^2 \\
&= E\left\{V_R^4 + V_I^4 + 2V_R^2 V_I^2\right\} - \left(E\left\{V_R^2 + V_I^2\right\}\right)^2 \\
&= 2P^2 - P^2 = P^2
\end{aligned}
$$

Signal and Noise Components Are Both Gaussian

Both the ionospheric signal and noise contributions to the received voltages are individually Gaussian random variables.

$$
\begin{aligned}
V &= V_S + V_N \\
E\left\{V_S V_N^*\right\} &= 0 \\
E\left\{\left|V_S\right|^2\right\} &= S \\
E\left\{\left|V_N\right|^2\right\} &= N \\
E\left\{\left|V\right|^2\right\} &= S + N = P
\end{aligned}
$$

This is unlike other types of radar problems where the signals are treated as deterministic quantities.

Power Estimation

Given K voltage samples with unknown signal power S, a known noise power N, and total power $P = S + N$, an estimate of the signal power is:

$$\hat{S} = \frac{1}{K} \sum_{n=0}^{K-1} |V_n|^2 - N$$

Expected Value: $E\left\{\hat{S}\right\} = \frac{1}{K}\sum_{n=0}^{K-1} E\left\{|V_n|^2\right\} - N = P - N = S$

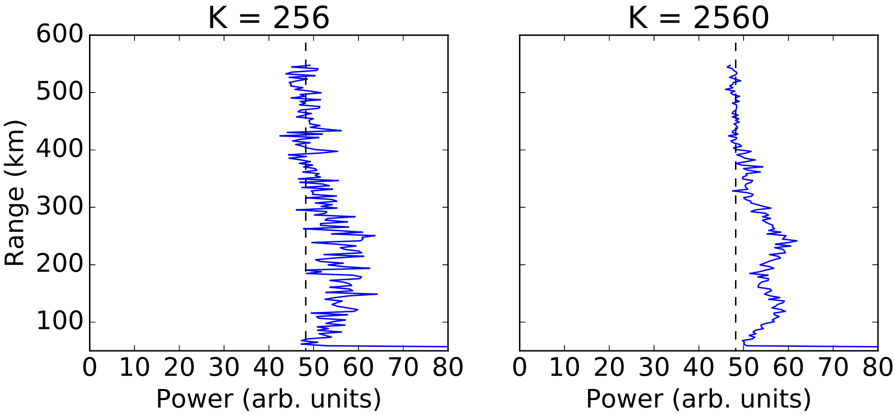
Variance (Invoke the Central Limit Theorem):

$$Var\left\{\hat{S}\right\} = Var\left\{\frac{1}{K}\sum_{n=0}^{K-1} |V_n|^2\right\} = \frac{1}{K} Var\left\{|V_n|^2\right\} = \frac{1}{K} P^2 = \frac{1}{K}(S+N)^2$$

Relative Error:

$$\frac{\sqrt{Var\left\{\hat{S}\right\}}}{S} = \frac{1}{\sqrt{K}} \frac{S+N}{S} = \frac{1}{\sqrt{K}}\left(1 + \frac{1}{S/N}\right)$$

Statistical Uncertainty and SNR are Different Concepts

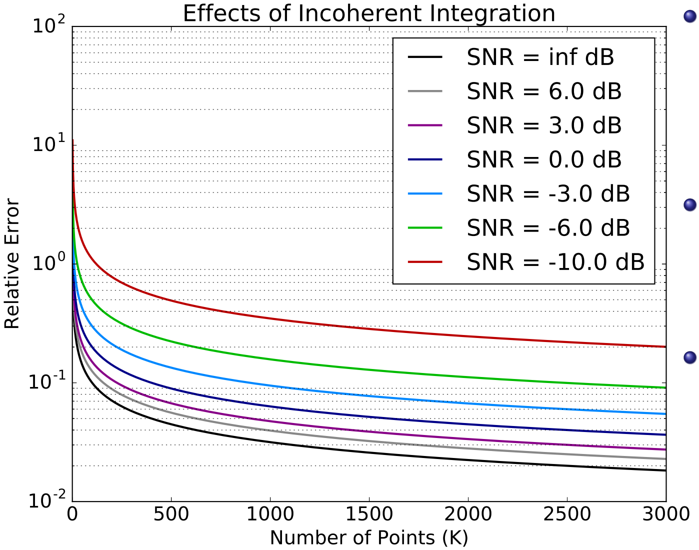


For $SNR = 0.25$:

$$K = 256 \rightarrow \text{Relative Error} = 31.25\%$$

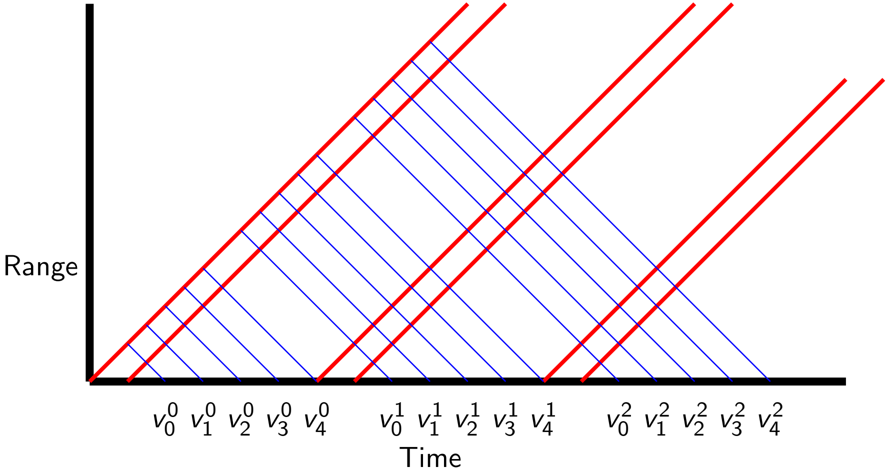
$$K = 2560 \rightarrow \text{Relative Error} = 9.88\%$$

Required Integration Times



- At $SNR = -3$ dB, 20% error requires $K = 225$.
- If the inter-pulse period is 5 ms, 225 pulses takes 1.125 s.
- If you cycle between 25 beams, 225 pulses in all beams takes 28.125 s.

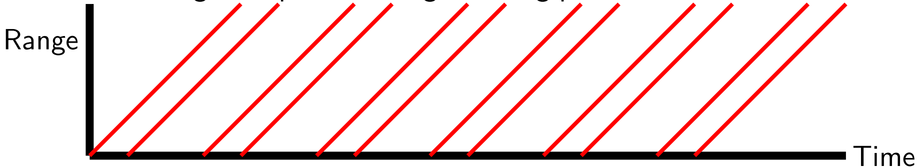
Problem with Short IPP: Range Aliasing



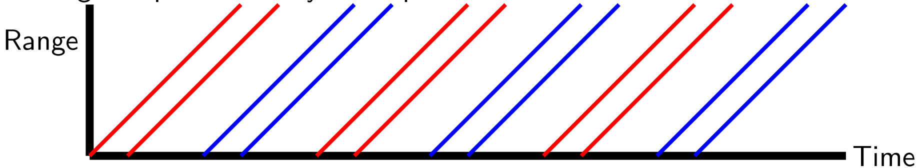
$$r_n = \frac{ct_n}{2} + m\frac{c\tau_{\mathrm{IPP}}}{2} \qquad \text{for any integer } m$$

Exploiting Frequency Diversity

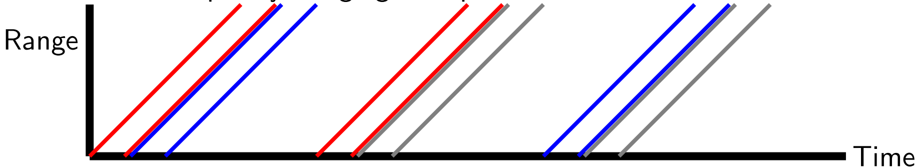
Pulses close together produce range aliasing problems:



Change frequencies every other pulse:



The RISR 3-frequency ImagingLP experiments:



Power Estimation Summary

- In ISR, both the signal and noise portions of the voltages are Gaussian random variables.
- The SNR is the ratio of the power in the signal and noise portions of the voltage.
- What actually matters for detectability is the relative error of our estimate of signal power, which depends on both SNR and the number of samples averaged together.

$$\frac{\sqrt{Var\left\{\hat{S}\right\}}}{S} = \frac{1}{\sqrt{K}}\left(1 + \frac{1}{S/N}\right)$$

- Some amount of averaging is always necessary, even in the $SNR \to \infty$ limit.
- For $SNR > 1$ it is more worthwhile to increase the effective number of samples than to keep increasing SNR further.