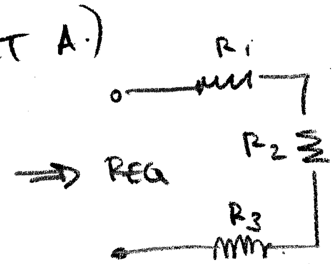
6.002 Fall 2007

Problem Set 1 Solutions:

Exercise 1.1

PART A.)

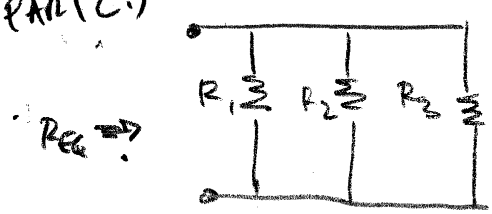


THREE SERIES RESISTORS:

$$\boxed{R_{EQ} = R_1 + R_2 + R_3.}$$

(see p. 78 + EQ 2.67 in A/L)

PART C.)

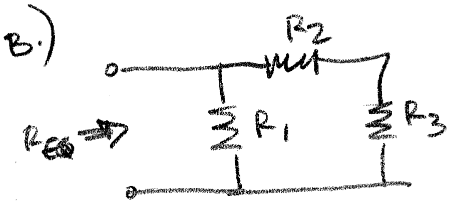


Three parallel resistors.

$$\frac{1}{R_{EQ}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

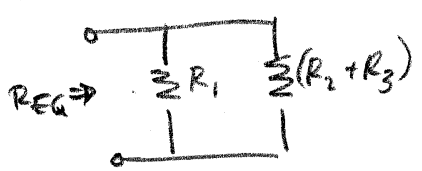
$$\boxed{R_{EQ} = \frac{R_1 R_2 R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3}}$$

PART B.)



R_1 in parallel w/ the series combination of $R_2 + R_3$.

simplify ↓



$$\frac{1}{R_{EQ}} = \frac{1}{R_1} + \frac{1}{R_2 + R_3}$$

$$\boxed{R_{EQ} = \frac{R_1 (R_2 + R_3)}{R_1 + R_2 + R_3}}$$

EXERCISE 1.2

the largest value equivalent resistor will be the series combination of the constituent resistors:

$$R_{SERIES} = 3\Omega + 3\Omega + 6\Omega = \boxed{12\Omega}$$

the smallest resistor will similarly be composed of the parallel combination of the constituent resistors:

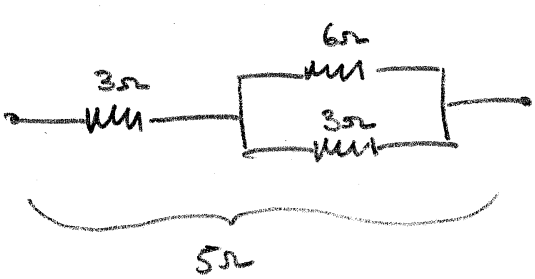
$$R_{parallel} = \frac{1}{\frac{1}{3}+\frac{1}{3}+\frac{1}{6}} = \boxed{6/5\,\Omega}$$

To fabricate a 5Ω resistor, we will need to put some resistors in parallel and some in parallel. The parallel combinations of two resistors yield

$$6\Omega \| 3\Omega = \frac{1}{\frac{1}{6}+\frac{1}{3}} = 2\Omega$$

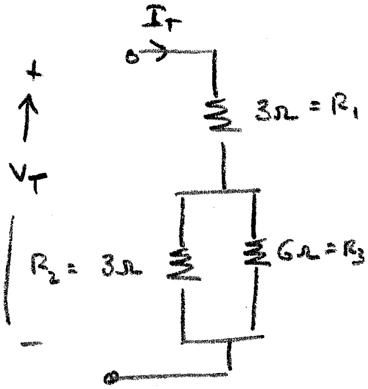
$$3\Omega \| 3\Omega = \frac{1}{\frac{1}{3}+\frac{1}{3}} = 3/2\,\Omega$$

Hm, a 2Ω resistor could be useful. So a 5Ω resistor could be made like:



Now let's look at the power consumption.

For the circuit under consideration, assume a voltage V_T is placed across the terminals. Then the total power consumed by the circuit is $VI_T = \frac{V_T^2}{R_{EQ}} = \frac{V_T^2}{5\Omega}$. Now calculate the power consumed by each element.



$$V_{R1} = \frac{R_1}{R_1 + (R_2 \| R_3)} V = 3/5 V_T$$

$$V_{R2} = V_{R3} = \frac{R_2 \| R_3}{R_1 + (R_2 \| R_3)} = 2/5 V_T$$

$$I_{R2} = \frac{V_{R2}}{3\Omega} = \frac{2}{15} V_T \quad ; \quad I_{R3} = \frac{V_{R3}}{6\Omega} = 1/15 V_T$$

$$I_{R1} = I_{R2} + I_{R3} = 3/15 V_T$$

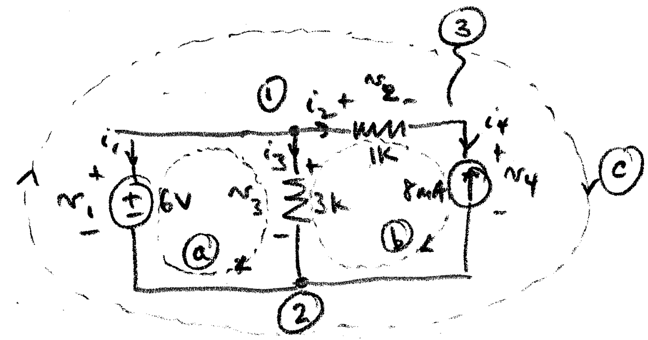
therefore $P_{R1} = V_{R1} I_{R1} = 9/75 V_T^2$; $P_{R2} = 4/75 V_T^2$; $P_{R3} = 2/75 V_T^2$

Note that $P_{R1} + P_{R2} + P_{R3} = 1/5 V_T^2$, as we calculated before.

If V_T^2 is such that $9/75 V_T^2 = 1W$, then we can calculate the total power by $V_T^2 = \frac{1W}{9/75} = \frac{xW}{\frac{15}{75}}$

so that $x = \frac{15}{9}$ W or $\boxed{1\tfrac{2}{3} W.}$

Problem 1.1



A.)

There are 3 nodes in this network. [1, 2, 3].

KCL @ 1: $-i_1 - i_2 - i_3 = 0$

KCL @ 2: $i_1 + i_3 + i_4 = 0$

KCL @ 3: $i_2 - i_4 = 0$.

These equations are not independent: their sum = 0.
But any 2 of the 3 are independent.

B.) There are also three loops:

KVL @ Loop a: $-v_1 + v_3 = 0$

KVL @ Loop b: $-v_3 + v_2 + v_4 = 0$

KVL @ Loop c: $-v_1 + v_2 + v_4 = 0$

These equations are also not independent, since the sum of KVL for loop a and KVL for loop b = KVL loop c.

c.) The v-i constitutive laws are

$$\left.\begin{aligned} v_1 &= 6V \\ v_2 &= (i_2)(1k\Omega) \\ v_3 &= (i_3)(3k\Omega) \\ i_4 &= -8\,mA. \end{aligned}\right\}$$

Note that the current thru the voltage source and the voltage across the current source are determined by the network, and not by element laws.

D.) substitute constitutive relations from (C.) into KCL at nodes ① & ③, and KVL @ loops ⓐ and ⓑ.

$$-i_1 - i_2 - i_3 = 0$$

$$i_2 - (-8mA) = 0 \Rightarrow i_2 = -8mA$$

$$-6V + v_3 = 0 \Rightarrow v_3 = 6V.$$

$$-v_3 + v_2 + v_4 = 0$$

Solving for the remainder of the variables by substitution:

$$v_2 = (-8mA)(1k\Omega) = -8V$$

$$v_4 = v_1 - v_2 = 6V + 8V = 14V$$

$$i_3 = v_3/3k = 6V/3k = 2mA$$

$$i_1 = -i_2 - i_3 = 8mA - 2mA = 6mA$$

i_1	i_2	i_3	i_4	v_1	v_2	v_3	v_4
6mA	-8mA	2mA	-8mA	6V	-8V	6V	14V

E.) checking to see that power is infact conserved:

$$v_1 i_1 = (6mA)(6V) = 36mW$$

$$v_2 i_2 = (-8mA)(-8V) = 64mW$$

$$v_3 i_3 = (2mA)(6V) = 12mW$$

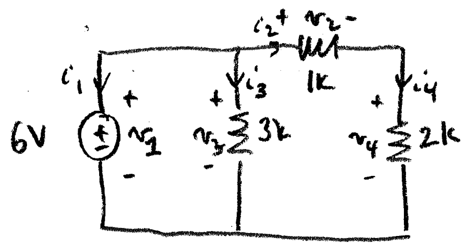
$$v_4 i_4 = (-8mA)(14V) = -112mW$$

$$\overline{\quad 0\ mW \checkmark}$$

since we used the convention of assigning polarities directed from (+) to (-) each positive $i_x v_x$ is a sink (representing power delivered into an element) and negative $v_x i_x$ is a power source.

Problem 1.2

Now the circuit looks like:



Let's start with what we can already do. Since we can write KVL for the loop including v_1 & v_3 : $-v_1+v_3=0$, we know that $v_3=6V$. We also know that $-v_3+v_2+v_4=0$, (or $-v_1+v_2+v_4=0$), so we can figure out the voltages v_2 and v_4.

$$-6V+v_2+v_4=0$$

$$i_2-i_4=0 \Rightarrow i_2=i_4=i_t$$

$$v_2=(1k)i_t\,;\quad v_4=(2k)i_t$$

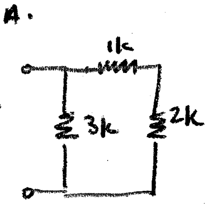
$$\Rightarrow -6V+(1k)i_t+(2k)i_t=0$$

$$\frac{6V}{3k}=i_t=2mA \quad \Rightarrow v_2=(1k)i_t=2V\,;\quad v_4=(2k)i_t=4V$$

We can also find $i_3=6V/3k=2mA$. Since we know that

$$-i_1-i_2-i_3=0\quad,\quad -i_1-2mA-2mA=0 \Rightarrow i_1=-4mA.$$

We could also have deduced this by computing $R_{EQ} \Rightarrow$



$$R_{EQ}=3k\,\|\,(1k+2k)=\frac{(3k)(3k)}{3k+3k}=1.5k.$$

$$i_1=-\frac{6V}{3/2\,k}=-4mA.$$ Either way,

i_1	i_2	i_3	i_4	v_1	v_2	v_3	v_4
−4mA	2mA	2mA	2mA	6V	2V	6V	4V

we could have also used a voltage divider to find v_2 & v_4. Since there is 6V across the series combination of $v_2 + v_4$,

$$v_2 = \frac{(1k)}{(1k+2k)}(6V) = 2V$$

$$v_4 = \frac{(2k)}{(2k+1k)}(6V) = 4V.$$

Problem 1.3

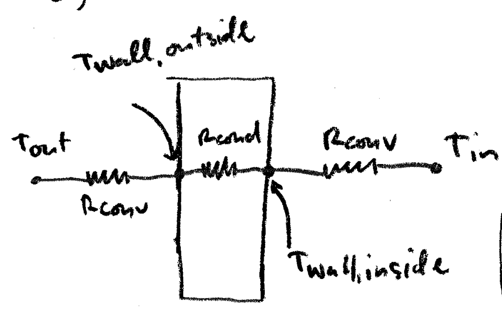
A.) For heat conduction, $Q = Ak\frac{\Delta T}{\Delta x}$. If ΔT is analogous to ΔV, and Q is analogous to I, where the relation between the two is $\Delta V = IR$, we want $\Delta T = Q R_{cond}$ ⇒

$$\boxed{R_{cond} = \frac{\Delta x}{Ak}}$$

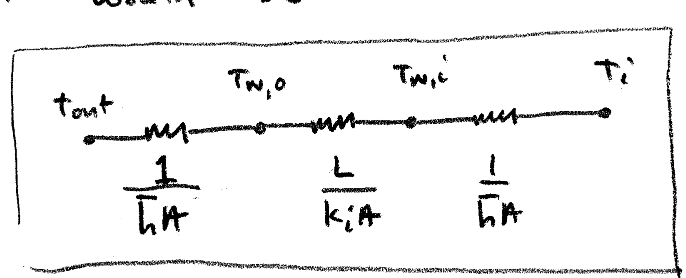
Similarly, for $Q = hA\Delta T$,

$$\boxed{R_{conv} = \frac{1}{hA}.}$$

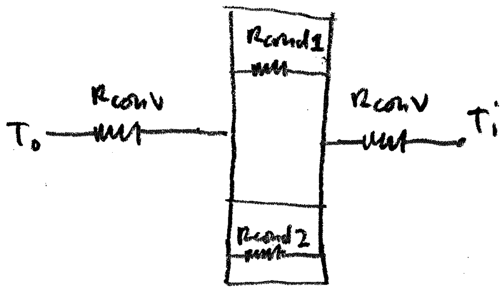
B.) For the wall construction of Figure 3a,



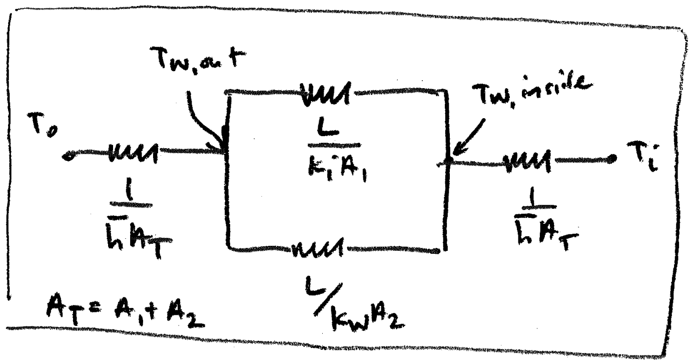
so the thermal model would be



c.) Similarly,



And the thermal circuit model would be



D.)

$$R_{conv,outside} = \frac{1}{h_{out} A} = \frac{1}{(30\,W/m^2k)(1m^2)} = 0.033\ K/W = 0.33\ ^\circ C/W$$

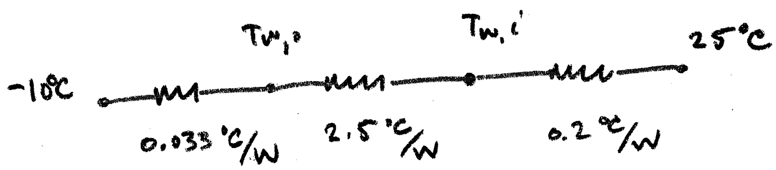
$$R_{conv,inside} = \frac{1}{h_{in} A} = \frac{1}{(5\,W/m^2k)(1m^2)} = 0.2\ K/W$$

$R_{cond,insulation}$ (with $0.8m^2$ area) $= \frac{L}{k_i A_1} = \frac{0.1m}{(0.04\,W/mk)(0.8m^2)} = 3.125\ K/W$

$$R_{cond,wood} = \frac{L}{k_w A_2} = \frac{0.1m}{(0.14\,W/mk)(0.2m^2)} = 3.571\ K/W$$

$R_{cond,insulation}$ (w/ $1m^2$ area) $= \frac{L}{k_i A_T} = \frac{0.1m}{(0.04\,W/mk)(1m^2)} = 2.5\ K/W$

The first wall model looks like:



(we can cavalierly change °C to K and vice versa here b/c we are looking at ΔT, not for thermodynamic properties. One must, in general, be careful of this → if we were working on radiative heat transfer, this wouldn't be correct.)

Anyhow,

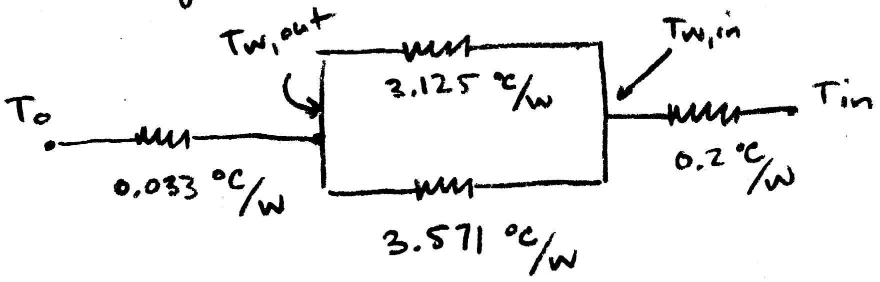
$$T_{w,i} = \frac{(2.5 + 0.033)\ ^\circ C/W}{(2.5 + 0.033 + 0.2)\ ^\circ C/W}\ \Delta T + T_o$$

$$= (0.927)(25^\circ C - (-10^\circ C)) - 10^\circ C$$

$$\boxed{T_{w,i} = 22.44^\circ C}$$

$$Q_{loss} = \frac{(25 - (-10))\ ^\circ C}{(2.5 + 0.033 + 0.2)\ ^\circ C/W} = \boxed{12.81 W = Q_{loss}}$$

The second model of the wall:



The parallel combination of the wall conductance terms =

$$\frac{(3.125\ ^\circ C/W)(3.571\ ^\circ C/W)}{(3.125 + 3.571)\ ^\circ C/W} = 1.667\ ^\circ C/W$$

so that

$$T_{w,i} = T_o + \frac{(1.667 + 0.033)\ ^\circ C/W}{(1.667 + 0.033 + 0.2)\ ^\circ C/W}\ \Delta T = \boxed{21.33^\circ C = T_{w,i}}$$

$$Q = \frac{(35^\circ C)}{(1.667 + 0.633 + 0.2)\ (^\circ C/W)} = \boxed{18.42 W = Q_{loss}}$$

On the basis of these numbers, the wall temperature is about the same, but the wall with the joists loses 50% more heat than the wall w/o the joists. Assuming that the insulation material was structurally useful, it would certainly be better to use the first type of wall construction. [Unfortunately, those wood joists tend to beat out fiberglass for structural stability.]. Naturally, the wall construction with the wood joists is probably a more accurate model; the model could probably be improved, since we are assuming perfect conductivity across the surface of the wall - there would more realistically be cold spots on the wall where the joists were located.

There could be many ways to improve the wall construction:

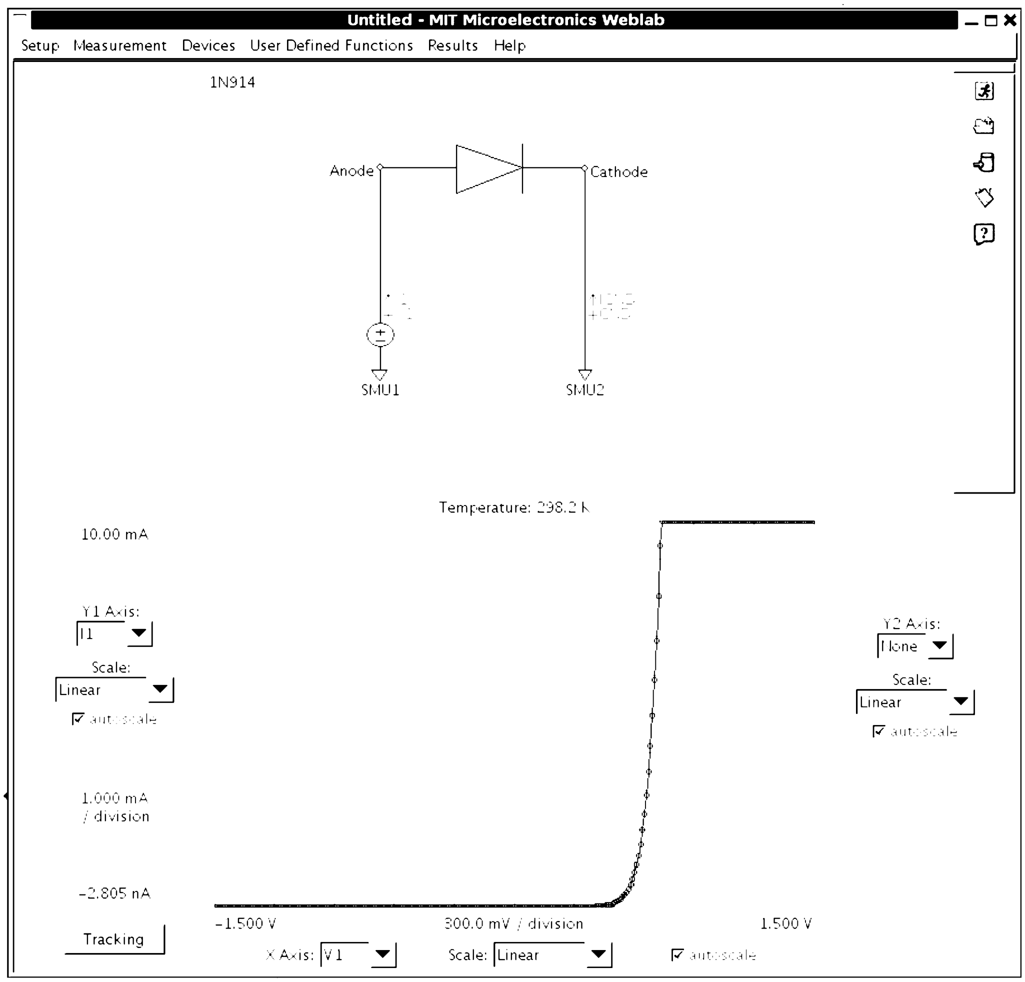
1.) Insulate next to the joists, so there is a layer of insulation btw. the room and the joists.

2.) Construct the joists out of lower conductivity material

3.) Make the surface area of the joists smaller

4.) &c.

Problem 4:



Parameters:

$i_D = I_S (e^{v_D/V_{TH}} - 1)$

V_T = thermal voltage (kT/q)

k = Boltzmann's const $= 1.78 \times 10^{-23}$ J/k

q = electron charge $= 1.60 \times 10^{-19}$ C

I_S = saturation current (A)

T = diode temperature (k)

v_D = diode voltage (V)

i_D = diode current (A)

Problem 4 (Continued):

A) From the WebLab,

$$V_T = \frac{kT}{q} = \frac{(1.38\times10^{-23} J/k)(298.2k)}{(1.60\times10^{-19} C)} = \boxed{0.02572\ V}$$

B.) The saturation current I_S is the lower limit of the current through the diode. On the logarithmic measurement graph (see next page), $i_D \simeq -I_S$ can be seen on the left side of the graph, where i_D is relatively constant.

From the WebLab graph, $v_D < -0.1V$ shows that i_D is reasonably constant @ $i_D \simeq 2\times10^{-9}A$.

$$\Rightarrow I_S \simeq 2\times10^{-9}A$$

Using the constitutive relation for the diode

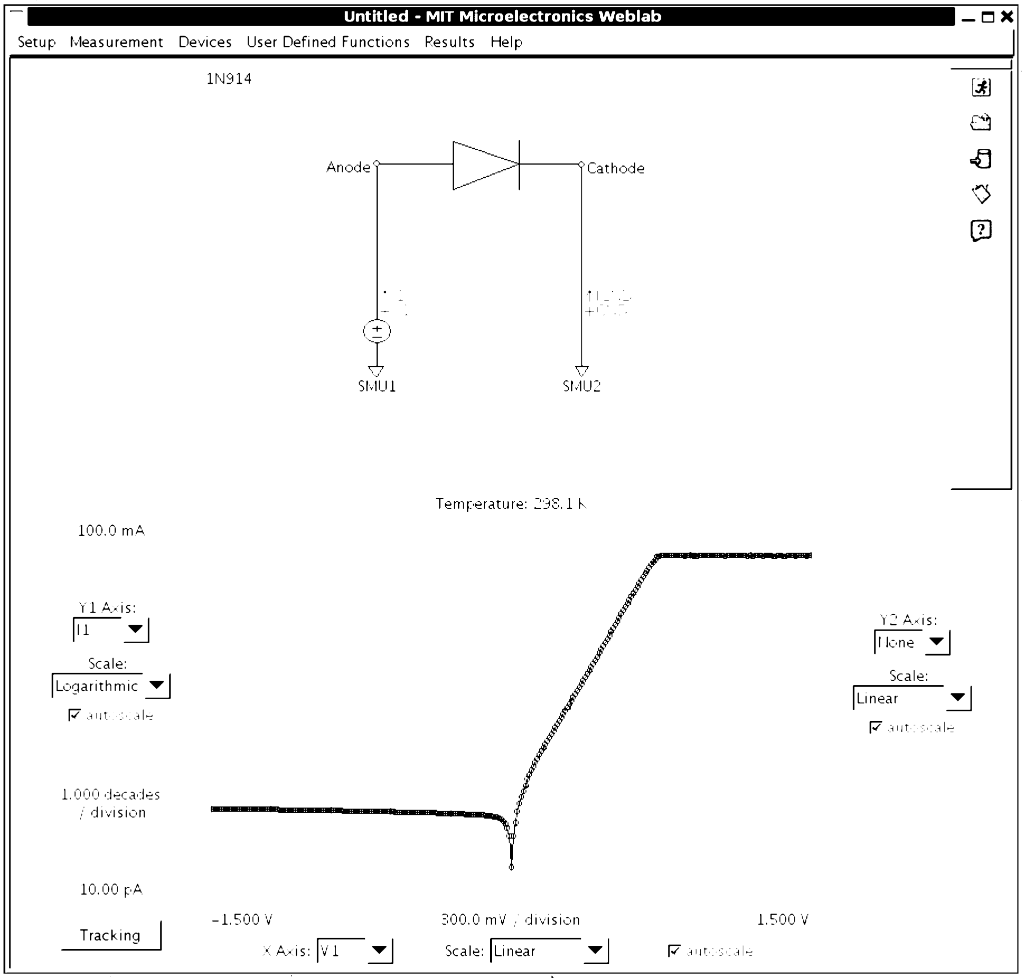
$i_D = I_S(e^{-v_D/V_T} - 1)$, $i_D \simeq -I_S$ when $e^{v_D/V_{TH}} \ll 1$

let's find the point that $e^{v_D/V_{TH}} < 0.01 \ll 1$

$$v_D < \ln(0.01)\cdot V_{TH}$$

$$\boxed{v_D < -0.11844}$$

So our estimate for I_S is fairly reasonable.



↑ (logarithmic diode characteristic)

c.) the portion of the curve that follows the exponential relation $i_D = I_S e^{v_D/V_{TH}}$. the WebLab graph indicates that this is where $v_D > 0.2V$

Using the constitutive relation, $i_D = I_S(e^{v_D/V_{TH}} - 1)$

$$i_D \simeq I_S e^{v_D/V_{TH}} \text{ when } (e^{v_D/V_{TH}}) \gg 1$$

$$\Rightarrow e^{v_D/V_{TH}} > 100$$

$$v_D > V_{TH} \ln(100)$$

$$v_D > 0.118\text{ V}$$