6.002 Spring 2008
Problem Set 4
Solutions

Exercise 4.1:

$$OUT = \overline{IN1 + \overline{(\overline{IN2} \cdot IN3)}}$$

IN1	IN2	IN3	OUT
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	0

De Morgan's Theorem

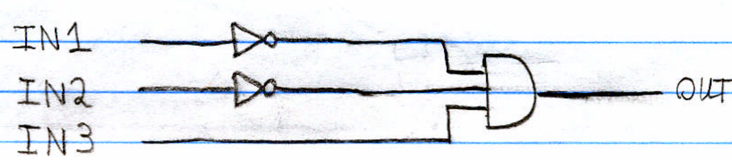
$$\overline{X + Y} = \overline{X} \cdot \overline{Y}$$

$$OUT = \overline{IN1 + \overline{(\overline{IN2} \cdot IN3)}}$$

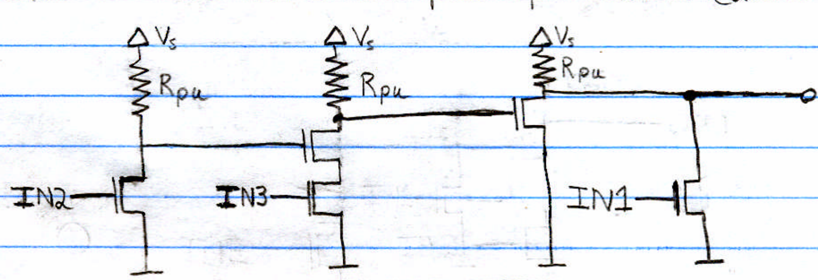
$$OUT = \overline{IN1} \cdot \overline{\overline{\overline{IN2} \cdot IN3}}$$

$$OUT = \overline{IN1} \cdot \overline{IN2} \cdot IN3$$

Using logic symbols:

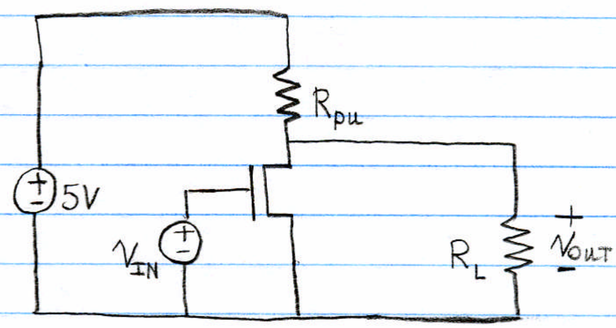


Using n-channel MOSFETs and pull-up resistors: (One of many possible ccts)



6.002 Pset 4
Solutions

Exercise 4.2:



$R_L \geq 8M\Omega$

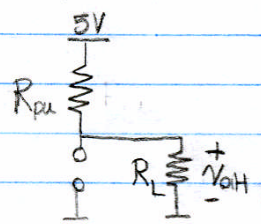
$V_{OL} = 1V, \; V_{IL} = 2V$

$V_{IH} = 3V, \; V_{OH} = 4V$

$V_{IL} < V_T < V_{IH}$

$\underline{2 < V_T < 3}$

$V_{IN} < V_T$



$$V_{OH} = 4 < \frac{R_L}{R_{pu} + R_L}(5V)$$

$$\frac{R_L}{R_{pu} + R_L} > \frac{4}{5} = 0.8$$

$$R_L > 0.8(R_{pu} + R_L)$$

$$0.2R_L > 0.8R_{pu}$$

$$R_L > 4R_{pu} \Rightarrow R_{pu} < \frac{1}{4}R_L$$

Given: $R_L \geq 8M\Omega$

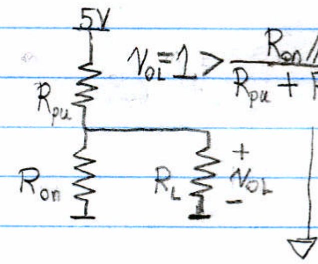
$\Rightarrow R_{pu} \leq 2M\Omega$

Need to minimize power dissipation so must maximize the resistance

$$P = \frac{V^2}{R} = \frac{(5)^2}{R_{pu} + R_L}$$

$\Rightarrow \underline{R_{pu} = 2M\Omega}$

$V_{IN} > V_T$



$$V_{OL} = 1 > \frac{R_{on} /\!/ R_L}{R_{pu} + R_{on} /\!/ R_L}(5V)$$

$$\frac{R_{on} /\!/ R_L}{R_{pu} + R_{on} /\!/ R_L} < \frac{1}{5} = 0.2$$

$$R_{on} /\!/ R_L < 0.2(R_{pu} + R_{on} /\!/ R_L)$$

$$0.8\, R_{on} /\!/ R_L < 0.2\, R_{pu}$$

$$R_{on} /\!/ R_L < \frac{1}{4} R_{pu}$$

$$R_{on} /\!/ R_L < 0.5M\Omega$$

$$\frac{R_{on} R_L}{R_{on} + R_L} < 0.5M\Omega$$

$$R_{on} R_L < 0.5M\Omega(R_{on} + R_L)$$

$$R_{on} < 0.5M\Omega\left(\frac{R_{on}}{R_L} + 1\right)$$

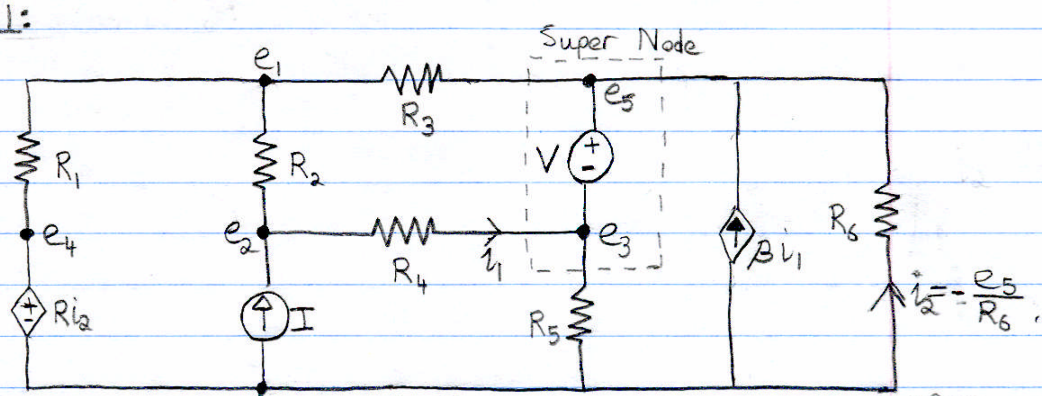
Worst case: $R_L \to \infty$

$\Rightarrow R_{on} < 0.5M\Omega$

$$P = \frac{V^2}{R} = \frac{(5)^2}{R_{pu} + R_{on} /\!/ R_L}$$

$\Rightarrow \underline{R_{on} = 0.5M\Omega}$

Problem 4.1:



$$i_1 = \frac{e_2 - e_3}{R_4}$$

$$e_4 = Ri_2 = R\left(-\frac{e_5}{R_6}\right) = -R\frac{(e_3+V)}{R_6}$$

$$e_5 = e_3 + V$$

KCL @ e_1:

$$\frac{e_1 - e_4}{R_1} + \frac{e_1 - e_2}{R_2} + \frac{e_1 - e_5}{R_3} = 0$$

$$\frac{e_1 + \frac{R}{R_6}(e_3+V)}{R_1} + \frac{e_1 - e_2}{R_2} + \frac{e_1 - e_3 - V}{R_3} = 0$$

$$\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)e_1 - \frac{1}{R_2}e_2 + \left(\frac{R}{R_1R_6} - \frac{1}{R_3}\right)e_3 = -\left(\frac{R}{R_1R_6} - \frac{1}{R_3}\right)V$$

KCL @ e_2:

$$\frac{e_2 - e_1}{R_2} + \frac{e_2 - e_3}{R_4} = I$$

$$-\frac{1}{R_2}e_1 + \left(\frac{1}{R_2} + \frac{1}{R_4}\right)e_2 - \frac{1}{R_4}e_3 = I$$

KCL @ e_3:

$$\frac{e_3 - e_2}{R_4} + \frac{e_3}{R_5} + \frac{e_5 - e_1}{R_3} + \frac{e_5}{R_6} - \beta i_1 = 0$$

$$\frac{e_3 - e_2}{R_4} + \frac{e_3}{R_5} + \frac{e_3 + V - e_1}{R_3} + \frac{e_3 + V}{R_6} - \beta\frac{(e_2 - e_3)}{R_4} = 0$$

$$-\frac{1}{R_3}e_1 - \left(\frac{1}{R_4} + \frac{\beta}{R_4}\right)e_2 + \left(\frac{1}{R_4} + \frac{1}{R_5} + \frac{1}{R_3} + \frac{1}{R_6} + \frac{\beta}{R_4}\right)e_3 = -\left(\frac{1}{R_3} + \frac{1}{R_6}\right)V$$

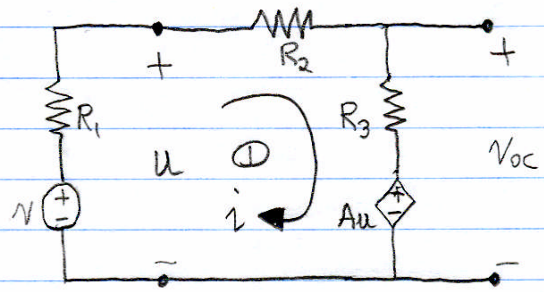
Problem 4.1: (Continued)

$$\begin{bmatrix} \frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3} & -\frac{1}{R_2} & \frac{R}{R_1 R_6}-\frac{1}{R_3} \\ -\frac{1}{R_2} & \frac{1}{R_2}+\frac{1}{R_4} & -\frac{1}{R_4} \\ -\frac{1}{R_3} & -\left(\frac{1}{R_4}+\frac{\beta}{R_4}\right) & \frac{1}{R_4}+\frac{1}{R_5}+\frac{1}{R_3}+\frac{1}{R_6}+\frac{\beta}{R_4} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{R_3}-\frac{R}{R_1 R_6}\right)V \\ I \\ -\left(\frac{1}{R_3}+\frac{1}{R_6}\right)V \end{bmatrix}$$

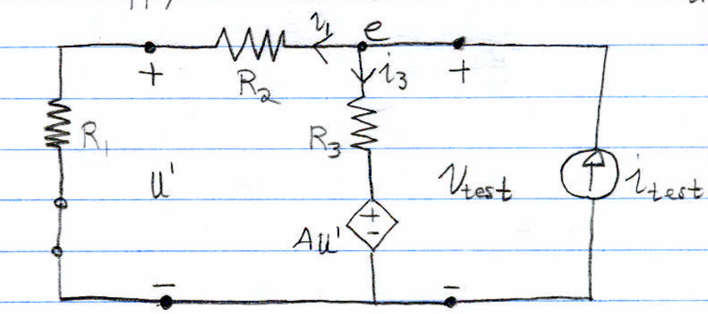
$$e_4 = -\frac{R}{R_6}(e_3+V)$$

Problem 4.2:

(A) Left cct.:



Turn off sources and apply a test current to find R_{th}:



$$R_{TH} = \frac{v_{test}}{i_{test}}$$

KCL @ node e: $e = v_{test}$

$$\frac{e}{R_1+R_2} + \frac{e - Au'}{R_3} = i_{test} \qquad u' = i_1 R_1 = \frac{v_{test}}{R_1+R_2} R_1$$

$$\frac{v_{test}}{R_1+R_2} + \frac{v_{test}}{R_3} - \frac{AR_1 v_{test}}{R_3(R_1+R_2)} = i_{test}$$

$$\left[\frac{1}{R_1+R_2} + \frac{1}{R_3} - \frac{AR_1}{R_3(R_1+R_2)}\right] v_{test} = i_{test}$$

$$\underline{R_{th} = \frac{v_{test}}{i_{test}} = \frac{R_3(R_1+R_2)}{R_3+R_1+R_2-AR_1}}$$

KVL around loop 1: $v = R_1 i + R_2 i + R_3 i + Au$ where $u = v - R_1 i$

$$v = R_1 i + R_2 i + R_3 i + Av - AR_1 i$$

$$(R_1+R_2+R_3-AR_1)\, i = (1-A)v$$

$$i = \frac{(1-A)v}{(1-A)R_1+R_2+R_3}$$

Problem 4.2: (Continued)

Calculate $V_{th} = v_{oc}$:

$$v_{oc} = Au + iR_3$$
$$= Av - AR_1 i + iR_3$$
$$= Av - (AR_1 - R_3)i$$

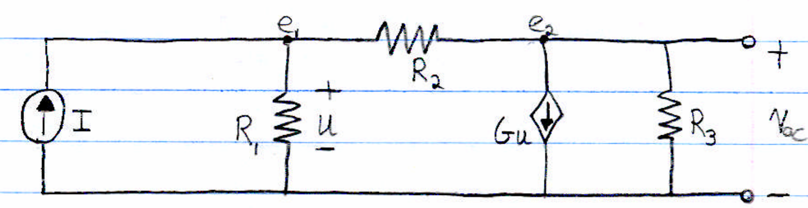
$$= Av - \frac{(AR_1 - R_3)(1-A)v}{(1-A)R_1 + R_2 + R_3}$$

$$= \frac{[(1-A)R_1 + R_2 + R_3]Av - (AR_1 - R_3)(1-A)v}{(1-A)R_1 + R_2 + R_3}$$

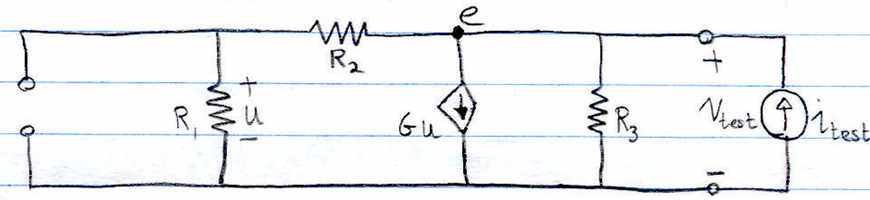
$$= \frac{\cancel{A(1-A)R_1 v} + AR_2 v + \cancel{AR_3 v} - \cancel{A(1-A)R_1 v} + R_3 v - \cancel{AR_3 v}}{(1-A)R_1 + R_2 + R_3}$$

$$\underline{V_{th} = v_{oc} = \frac{(AR_2 + R_3)v}{(1-A)R_1 + R_2 + R_3}}$$

(B) Right cct.:

① 

Turn off sources and apply a test current to find R_{th}:



$$R_{TH} = \frac{v_{test}}{i_{test}}$$

KCL @ node e:

$$\frac{e}{R_1 + R_2} + Gu + \frac{e}{R_3} = i_{test}$$

$e = v_{test}$

$u = \frac{R_1}{R_1 + R_2} e$

$$\frac{v_{test}}{R_1 + R_2} + \frac{GR_1 v_{test}}{R_1 + R_2} + \frac{v_{test}}{R_3} = i_{test}$$

$$\left[\frac{1 + GR_1}{R_1 + R_2} + \frac{1}{R_3}\right] v_{test} = i_{test}$$

Problem 4.2: (Continued)

$$R_{TH} = \frac{v_{test}}{i_{test}} = \frac{(R_1 + R_2)R_3}{(1 + GR_1)R_3 + R_1 + R_2}$$

Calculate $V_{th} = v_{oc}$:

KCL @ node e_1 in cct ①:

$$\frac{e_1}{R_1} + \frac{e_1 - e_2}{R_2} = I$$

$$\left(\frac{1}{R_1} + \frac{1}{R_2}\right)e_1 - \frac{1}{R_2}e_2 = I$$

$$\frac{R_1 + R_2}{R_1 R_2} e_1 = I + \frac{e_2}{R_2}$$

$$e_1 = \frac{R_1 R_2}{R_1 + R_2}\left[I + \frac{e_2}{R_2}\right]$$

KCL @ node e_2 in cct ①:

$$\frac{e_2 - e_1}{R_2} + Gu + \frac{e_2}{R_3} = 0 \qquad u = e_1$$

$$\left[\frac{1}{R_2} + \frac{1}{R_3}\right]e_2 + \left[G - \frac{1}{R_2}\right]e_1 = 0$$

$$\left[\frac{R_3 + R_2}{R_2 R_3}\right]e_2 + \left[G - \frac{1}{R_2}\right]\left[\frac{R_1 R_2}{R_1 + R_2}\right]\left[I + \frac{e_2}{R_2}\right] = 0$$

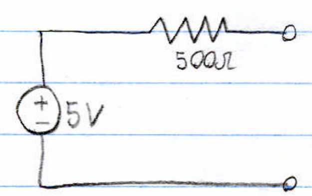
$$\left[\frac{R_2 + R_3}{R_2 R_3} + \left[G - \frac{1}{R_2}\right]\left[\frac{R_1 R_2}{R_1 + R_2}\right]\frac{1}{R_2}\right]e_2 = -\left[G - \frac{1}{R_2}\right]\left[\frac{R_1 R_2}{R_1 + R_2}\right]I$$

$$V_{TH} = v_{oc} = e_2 = \frac{\left[\frac{1}{R_2} - G\right]\left[\frac{R_1 R_2}{R_1 + R_2}\right]I}{\frac{R_2 + R_3}{R_2 R_3} + \left[G - \frac{1}{R_2}\right]\left[\frac{R_1 R_2}{R_1 + R_2}\right]\frac{1}{R_2}}$$

Problem 4.3:

(a) See page 9.

(b)



Looks like Thevenin equivalent circuit where

$V_{TH} = 5\,V$

$R_{TH} = 500\,\Omega$

$I_N = \frac{5}{500} = 0.01\,A$

See page 9 for $v_{DS} - i_D$ load line.

Table of resulting input-output relation:

$v_{IN} = v_{GS}(V)$	2.50	2.45	2.40	2.35	2.30	2.25	2.20	2.15	2.10
$v_{OUT} = v_{DS}(V)$	0.40	1.35	2.35	3.20	3.75	4.20	4.50	4.70	4.80

(c) See page 10.

(d) See page 10. The input-output relations found in Parts (B) and (C) are identical, as expected.

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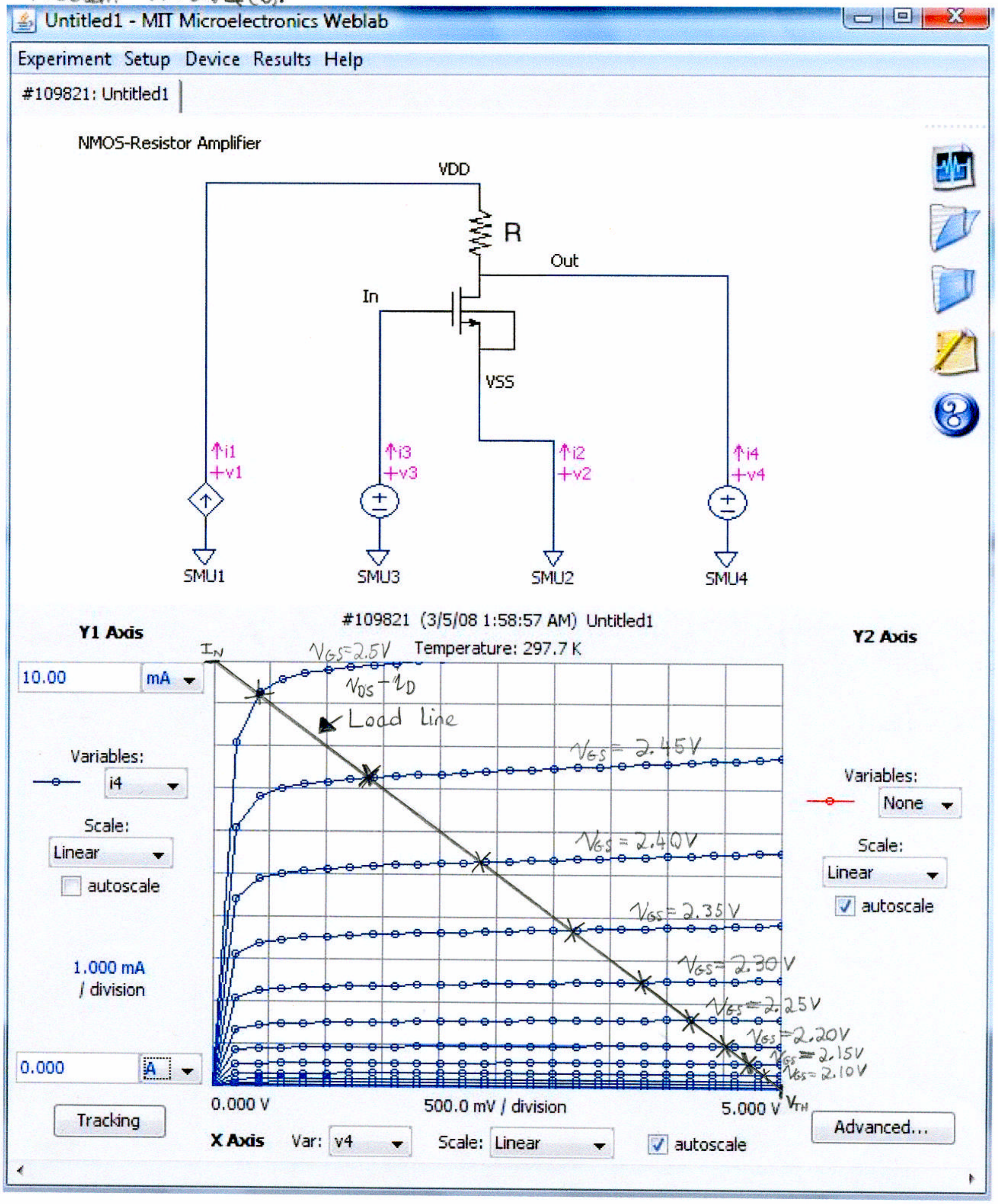
Solutions

Problem 4.3(a) & (b):



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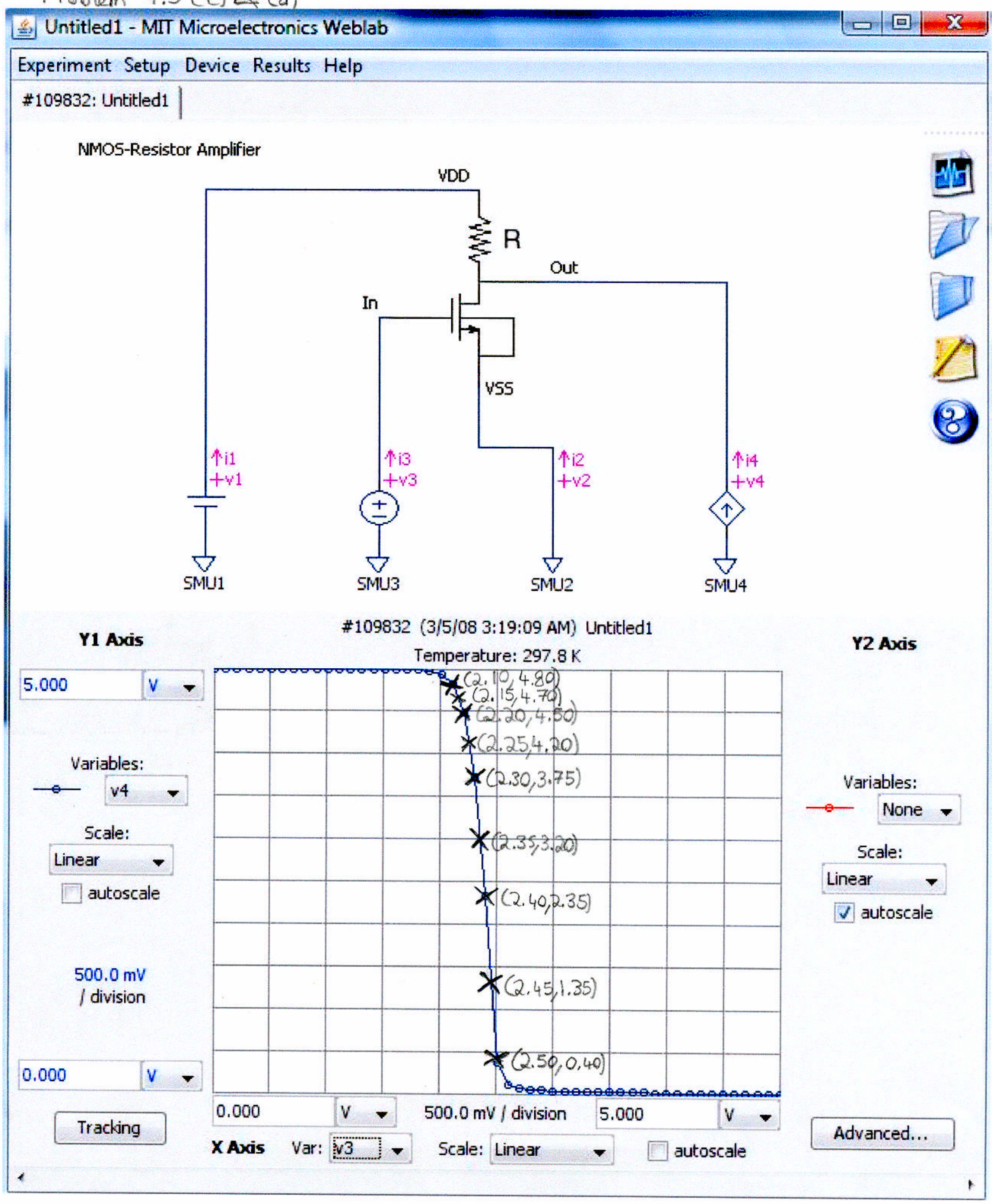
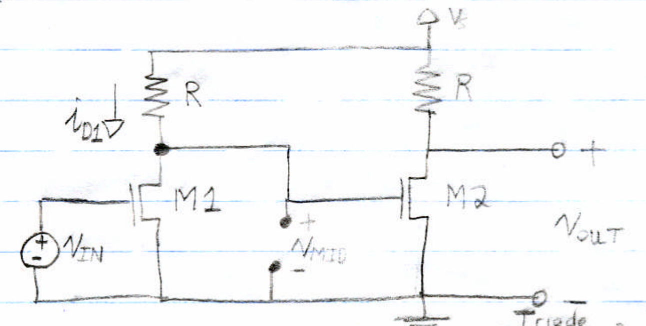
Problem 4.3 (c) & (d)



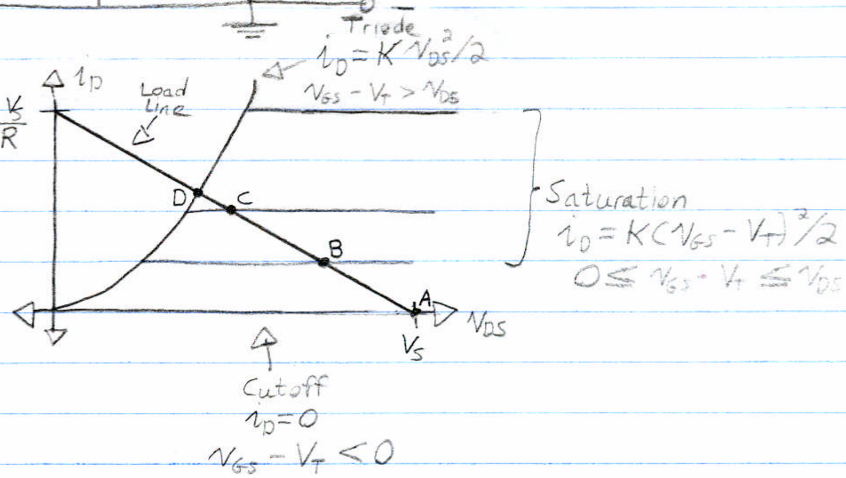
Problem 4.4:

Given:

①



②



(a) M1 will be in cutoff when $v_{IN} < V_T$. In this range $i_{D1} = 0$ so $v_{DS1} = v_{MID} = V_S$

(b) Assuming M1 in saturation ($v_{IN} > V_T$) then a current i_{D1} will be flowing through pull-up resistor and M1.
So $v_{MID} = V_S - i_{D1}R$ where $i_{D1} = \frac{K}{2}(v_{IN} - V_T)^2$ (M1 in saturation)

$$\Rightarrow v_{MID} = V_S - \frac{RK}{2}(v_{IN} - V_T)^2$$

To determine the range of v_{MID} and the range of v_{IN} that correspond to the saturated operation of M1 need to perform load line analysis, see load line on figure ② above. At point A M1 is in cutoff. As $v_{IN} = v_{GS1}$ is increased, it moves up the load line into saturation (point B). It will remain in saturation as v_{IN} is increased a bit more (point C), until it hits

Problem 4.4: (Continued)

(B) the triode boundary (point D). So to find ranges of v_{IN}, v_{MID} of saturation need to find point D.

Can find point D by equating load line to triode/saturation boundary:

$$i_{D1} = \frac{K}{2} v_{MID,D}^2 = \frac{V_S}{R} - \frac{v_{MID,D}}{R}$$

$$\frac{K}{2} v_{MID,D}^2 + \frac{v_{MID,D}}{R} - \frac{V_S}{R} = 0$$

← Quadratic Equation
$a = \frac{K}{2}$
$b = \frac{1}{R}$
$c = -\frac{V_S}{R}$

$$v_{MID,D} = \frac{-\left(\frac{1}{R}\right) \pm \sqrt{\left(\frac{1}{R}\right)^2 - 4\left(\frac{K}{2}\right)\left(-\frac{V_S}{R}\right)}}{2\left(\frac{K}{2}\right)}$$

$$v_{MID,D} = \frac{1}{RK}\left(-1 + \sqrt{1 + 2RKV_S}\right)$$

Pick positive solution ($v_{MID,D}$ is positive). This is lower bound for $v_{MID,D}$ to still be in saturation.

The corresponding $v_{IN,D}$ can be found at the triode/saturation boundary:

$$v_{GS} - V_T = v_{DS} = v_{MID,D}$$

$$v_{IN,D} - V_T = v_{MID,D}$$

$$v_{IN,D} = v_{MID,D} + V_T$$

Max v_{IN} can go is V_T above $v_{MID,D}$

The other boundary to saturation is cutoff: $v_{IN} > V_T$. Therefore the ranges of v_{MID} and v_{IN} in saturation:

$$\frac{1}{RK}\left(\sqrt{1 + 2RKV_S} - 1\right) \leq v_{MID} \leq V_S$$

$$V_T \leq v_{IN} \leq \frac{1}{RK}\left(\sqrt{1 + 2RKV_S} - 1\right) + V_T$$

Problem 4.4: (Continued)

(C) If v_{IN} is increased beyond limits stated in part B then the operating point stays at point D. It can't go up along triode curve because it must intersect with load line. Thus for

$$v_{IN} \geq \frac{1}{RK}\left(\sqrt{1+2RKV_S}-1\right)+V_T$$
$$v_{MID} = v_{MID,D} = \frac{1}{RK}\left(\sqrt{1+2RKV_S}-1\right)$$

(D) In the case of M2 $v_{GS2} = v_{MID}$ (output of M1 is input of M2). Since R's and MOSFETs are identical the v_{OUT} and v_{MID} relationship is the same as that of v_{MID} and v_{IN}.

M2 in cuttoff: $v_{MID} < V_T$, $v_{OUT} = V_S$

M2 in saturation:
$$V_T \leq v_{MID} \leq \frac{1}{RK}\left(\sqrt{1+2RKV_S}-1\right)+V_T$$
$$\frac{1}{RK}\left(\sqrt{1+2RKV_S}-1\right) \leq v_{OUT} \leq V_S$$

M2 in triode:
$$v_{MID} \geq \frac{1}{RK}\left(\sqrt{1+2RKV_S}-1\right)+V_T$$
$$v_{OUT} = \frac{1}{RK}\left(\sqrt{1+2RKV_S}-1\right)$$

(E) Assuming that both M1 and M2 operate in saturation From part B and D it is possible to determine v_{OUT} as a function of v_{IN}:

From part D: $v_{OUT} = V_S - \frac{RK}{2}(v_{MID}-V_T)^2$

From part B: $v_{MID} = V_S - \frac{RK}{2}(v_{IN}-V_T)^2$

$$\Rightarrow v_{OUT} = V_S - \frac{RK}{2}\left(V_S - \frac{RK}{2}(v_{IN}-V_T)^2 - V_T\right)^2$$

Problem 4.4: (Continued)

(E) To find the range of v_{MID}:

From part B: $\frac{1}{RK}(\sqrt{1+2RKV_S}-1) \le v_{MID} \le V_S$

From part D: $V_T \le V_{MID} \le \frac{1}{RK}(\sqrt{1+2RKV_S}-1)+V_T$

$$\Rightarrow \text{MAX}\{V_T, \frac{1}{RK}(\sqrt{1+2RKV_S}-1)\} \le v_{MID} \le \text{MIN}\{V_S, (\frac{1}{RK}(\sqrt{1+2RKV_S}-1)+V_T)\}$$

To find the range of v_{IN}:

From part B: $v_{MID} = V_S - \frac{RK}{2}(v_{IN}-V_T)^2$

$(v_{IN}-V_T)^2 = \frac{2}{RK}(V_S - v_{MID})$

$\Rightarrow v_{IN} = V_T + \sqrt{\frac{2}{RK}(V_S - v_{MID})}$

$v_{MID,MIN} \rightarrow v_{IN,MAX}$
$v_{MID,MAX} \rightarrow v_{IN,MIN}$

$$v_{IN,MIN} \le v_{IN} \le v_{IN,MAX}$$

where $v_{IN,MIN} = V_T + \sqrt{\frac{2}{RK}(V_S - \text{MIN}\{V_S, (\frac{1}{RK}(\sqrt{1+2RKV_S}-1)+V_T)\})}$

$v_{IN,MAX} = V_T + \sqrt{\frac{2}{RK}(V_S - \text{MAX}\{V_T, (\frac{1}{RK}(\sqrt{1+2RKV_S}-1))\})}$

(F) To find the small-signal gain $= \frac{d v_{OUT}}{d v_{IN}}$:

From part E: $v_{OUT} = V_S - \frac{RK}{2}(V_S - \frac{RK}{2}(v_{IN}-V_T)^2 - V_T)^2$

$$\frac{d v_{OUT}}{d v_{IN}} = -\frac{RK}{2}(2)(V_S - \frac{RK}{2}(v_{IN}-V_T)^2 - V_T) \cdot (-\frac{RK}{2}(2)(v_{IN}-V_T))$$

$$\frac{d v_{OUT}}{d v_{IN}} = (RK)^2(V_S - \frac{RK}{2}(v_{IN}-V_T)^2 - V_T)(v_{IN}-V_T)$$

Since: $v_{MID} = V_S - \frac{RK}{2}(v_{IN}-V_T)^2$

$$\frac{d v_{OUT}}{d v_{IN}} = (RK)^2(v_{MID}-V_T)(v_{IN}-V_T)$$

Problem 4.4: (Continued)

(G) Let:
$K = 0.01\,A/V^2$
$R = 2\,K\Omega$
$V_S = 10V$
$V_T = 1V$
$0 \le v_{IN} \le 3V$

① M1 Cutoff:
$0 \le v_{GS} \le V_T$
$0 \le v_{IN} \le 1 \qquad i_{D1} = 0$

② M1 Saturation:
$V_T \le v_{GS} \le \frac{1}{RK}\left(\sqrt{1+2RKV_S} - 1\right) + V_T$
$1 \le v_{IN} \le \frac{1}{(2000)(0.01)}\left(\sqrt{1+2(2000)(0.01)(10)} - 1\right) + 1$
$1 \le v_{IN} \le 1.95\,V$

$i_{D1} = \frac{K}{2}(v_{IN} - V_T)^2$
$i_{D1} = 0.005(v_{IN} - V_T)^2$

③ M1 Triode:
$v_{IN} \ge v_{IN,D}$
$v_{IN} \ge 1.95\,V$

$i_{D1} = i_{D1,D} = \frac{K}{2}(v_{IN,D} - V_T)^2$
$i_{D1} = 0.005(1.95 - 1)^2 = 4.5\,mA$

v_{MID} is obtained from: $v_{MID} = V_S - i_{D1}R$
Steps ①-③ can be repeated with $v_{IN} = v_{MID}$ to obtain equations for M2.
A MATLAB script was used to plot v_{MID} and v_{OUT} versus v_{IN} which is shown on the next page.

Problem 4.4: (Continued)

(G)