6.002 Problem set #5 - Solution

Exercise 5.1

MOSFET in saturation ∴ $V_S > v_{IN} - v_{OUT} - V_T$

$$i_D = \frac{K}{2}\left(v_{IN} - v_{OUT} - V_T\right)^2$$

$$v_{OUT} = R\, i_D = \frac{KR}{2}\left(v_{IN} - v_{OUT} - V_T\right)^2$$

Expanding to solve for v_{OUT}

$$v_{OUT} = \frac{KR}{2}\left[(v_{IN} - V_T)^2 - 2(v_{IN} - V_T)v_{OUT} + v_{OUT}^2\right]$$

$$\frac{KR}{2}v_{OUT}^2 - \left[KR(v_{IN} - V_T) + 1\right]v_{OUT} + \frac{KR}{2}(v_{IN} - V_T)^2 = 0$$

Solve using the quadratic formula,

$$v_{OUT} = \frac{KR(v_{IN} - V_T) + 1 \pm \sqrt{2KR(v_{IN} - V_T) + 1}}{KR}$$

$$v_{OUT} = (v_{IN} - V_T) + \frac{1}{KR} \pm \sqrt{\frac{2(v_{IN} - V_T)}{KR} + \frac{1}{(KR)^2}}$$

For MOSFET in saturation,

$$0 \le v_{GS} - V_T \le v_{DS}$$

$$0 \le v_{IN} - v_{OUT} - V_T \le V_S \quad (*)$$

$$\boxed{v_{IN} - V_T - V_S \le v_{OUT} \le v_{IN} - V_T} \Rightarrow \text{we choose the negative sign in the solution for } v_{OUT} \text{ above.}$$

$$\boxed{v_{OUT} = (v_{IN} - V_T) + \frac{1}{KR} - \sqrt{\frac{2(v_{IN} - V_T)}{KR} + \frac{1}{(KR)^2}}}$$

From (*), $v_{IN} \le v_{OUT} + V_T + V_S$

$$0 \le \cancel{v_{IN}} - (\cancel{v_{IN}} - \cancel{V_T}) + \frac{1}{KR} - \sqrt{\frac{2(v_{IN} - V_T)}{KR} + \frac{1}{(KR)^2}} - \cancel{V_T} \le V_S.$$

$$\therefore \boxed{0 \le \frac{1}{KR} - \sqrt{\frac{2(v_{IN} - V_T)}{KR} + \frac{1}{(KR)^2}} \le V_S}$$

For MOSFET to be in saturation.

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Exercise 5.2

M1 : $i_D = \frac{k_1}{2}(v_{IN} - V_{T1})^2$

M2: $i_D = \frac{k_2}{2}(V_S - v_{OUT} - V_{T2})^2$

Because of the circuit configuration, these two currents are equal.

$$\frac{k_1}{\cancel{2}}(v_{IN} - V_{T1})^2 = \frac{k_2}{\cancel{2}}(V_S - v_{OUT} - V_{T2})^2$$

$$\sqrt{\frac{k_1}{k_2}}(v_{IN} - V_{T1}) = V_S - v_{OUT} - V_{T2} .$$

$$\boxed{\therefore v_{OUT} = -\sqrt{\frac{k_1}{k_2}}(v_{IN} - V_{T1}) + V_S - V_{T2} .}$$

Note that this is more linear than the expression in E5.1

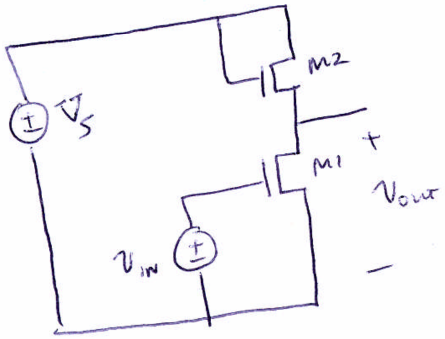
For M1 to be in saturation, $\boxed{0 \le v_{IN} - V_{T1} \le v_{OUT}}$ $\Rightarrow v_{IN} \ge V_{T1}$ and $v_{IN} \le v_{OUT} + V_{T1}$

For M2 to be in saturation,

$$0 \le V_S - v_{OUT} - V_{T2} \le V_S - v_{OUT} .$$

$\Rightarrow \boxed{V_S - v_{OUT} \ge V_{T2}}$ and $-V_{T2} \le 0$ (always true).

Recall:
In general, M is in saturation if $0 \le v_{GS} - V_T \le v_{DS}$.



M2 ⇒ "diode-connected" MOSFET. i.e. drain and gate are connected. The condition for M2 to be in saturation is less stringent than for M1 because M2 is diode-connected.

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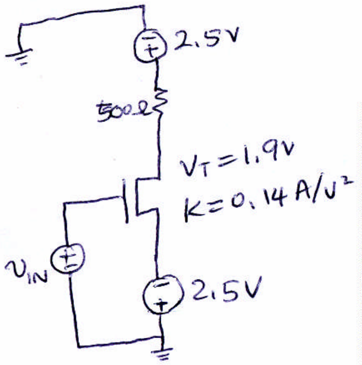
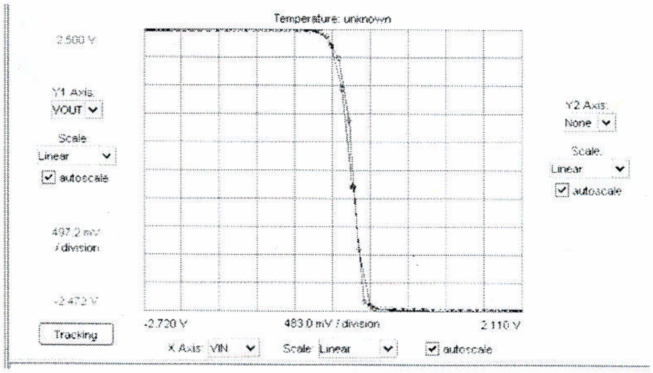
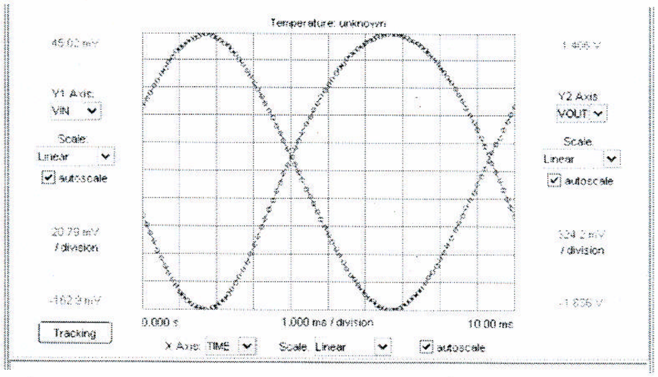


Problem 5.1 (A)



Using Tracking, we measure the input bias voltage of <u>83 mV</u> which gives V_{out} close to zero.

Problem 5.1 (B)



With an offset of <u>30 mV</u> on the input, we get a small-signal gain of

$$A_{ss} = \frac{-3.242}{208\ m} = \underline{\underline{-16\ V/V}}$$ as shown above.

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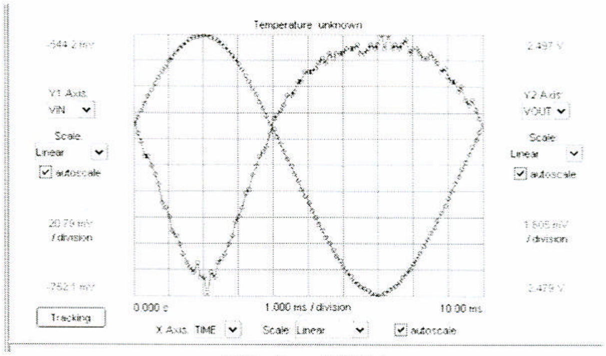
Problem 5.1 (C)



Y1 Axis: VIN
Scale: Linear
☑ autoscale
Y2 Axis: VOUT
Scale: Linear
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X Axis: TIME
Scale: Linear
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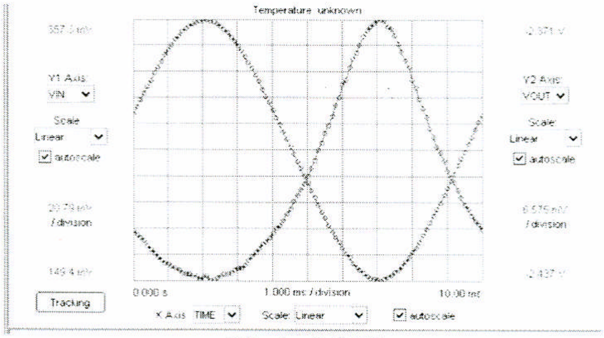
$A_{ss} = -8.8 \times 10^{-6}$ V/V

Offset = - 0.56V



Y1 Axis: VIN
Scale: Linear
☑ autoscale
Y2 Axis: VOUT
Scale: Linear
☑ autoscale
X Axis: TIME
Scale: Linear
☑ autoscale

$A_{ss} = -0.3175$ V/V

Offset = 0.34 V

For an offset of $\underline{-0.56V}$, we notice some distortion and greatly reduced gain. In this case,

$v_{IN} = -0.56V \quad \therefore v_{GS} = v_{IN} + 2.5V = 1.9V.$

Since $V_T = 1.9V$, the transistor is in cutoff, so the output current is very low. The distortion is less for the upper-half of the sine wave as expected.

For an offset of $\underline{0.34V}$, the gain is also greatly reduced. In this case

$v_{IN} = 0.34V \quad \therefore v_{GS} = v_{IN} + 2.5V = 2.84V$

For saturation, we require,

$v_{GS} - V_T < v_{DS}$

but in our case, $v_{GS} - V_T = 2.84 - 1.9 = 0.94V$ and

$v_{DS} = v_{OUT} + 2.5 = -2.4V + 2.5 = 100mV.$

so the transistor is not in saturation and has entered the triode region of operation

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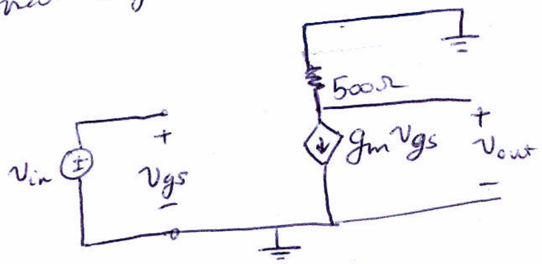
Problem 5.1 (D)

Small-signal model of our amplifier: We set all voltage sources as short circuits except for the small-signal input.



where $g_m = K(V_{GS} - V_T)$.

$$= 0.14(V_{GS} - V_T)$$

$$= 0.14(30\text{mV} - (-2.5\text{V}) - 1.9\text{V})$$

$$= 0.084 \text{ A/V}$$

Small-signal gain given by

$$A_{SS} = \frac{v_{out}}{v_{in}} = \frac{-g_m R\, v_{gs}}{v_{gs}} = -g_m R$$

$$\therefore A_{SS} = -0.084(500) = \boxed{-42 \text{ V/V}}$$

Note that this is larger than the small-signal gain measured in part (B). The difference is likely due to the fact that k is less than 0.14 A/V which can be seen by measuring the slope of the graph in part (A) and noting that

$$g_m = K(v_{GS} - V_T).$$

From this measurement, we get a value of k closer to 0.05 A/V. Using this value we get an expected gain of 15.75 V/V which is much closer to the value we obtained.

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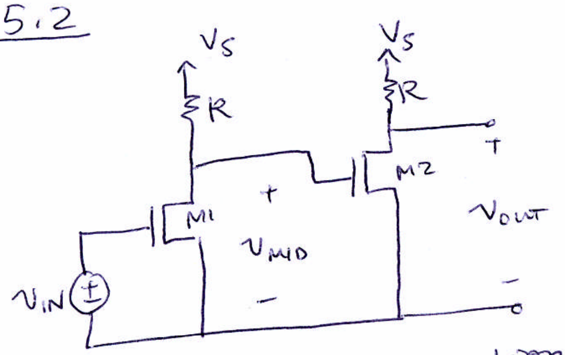
Problem 5.2



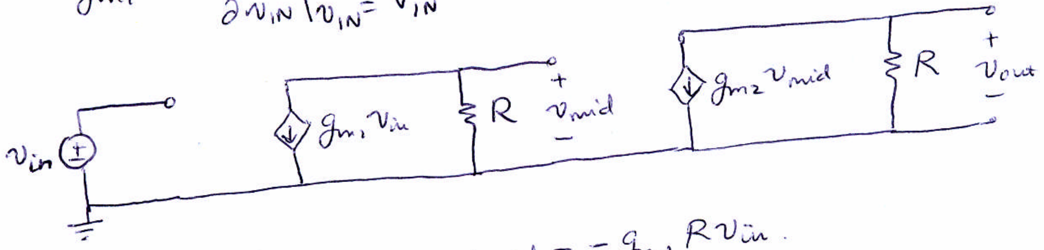
(A) Both MOSFETs are in saturation,

$$v_{MID} = V_S - i_D R = V_S - \frac{RK}{2}(v_{IN} - V_T)^2$$

$$i_{D2} = \frac{K}{2}(v_{MID} - V_T)^2 = \frac{K}{2}\left(V_S - \frac{RK}{2}(v_{IN} - V_T)^2 - V_T\right)^2$$

$$g_{m2} = \left.\frac{\partial i_{D2}}{\partial v_{MID}}\right|_{v_{MID}=V_{MID}} = K(V_{MID} - V_T) = K\left(V_S - \frac{RK}{2}(V_{IN} - V_T)^2 - V_T\right)$$

$$g_{m1} = \left.\frac{\partial i_{D1}}{\partial v_{IN}}\right|_{v_{IN}=V_{IN}} = K(V_{IN} - V_T)$$



$v_{mid} = -g_{m1} R v_{in}$.

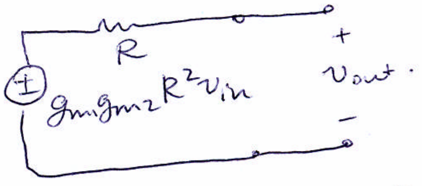
(B) $v_{out} = -g_{m2} R v_{mid}$

∴ Small-signal voltage gain:

$$\boxed{\frac{v_{out}}{v_{in}} = g_{m1} g_{m2} R^2}$$

(C) The gain expression given above is identical to that found in problem 4.3.

(D) To determine the Thevenin resistance, we set all independent sources to zero and find that we get resistor R as shown in the diagram above. The open-circuit voltage was found in (B), so our Thevenin circuit is



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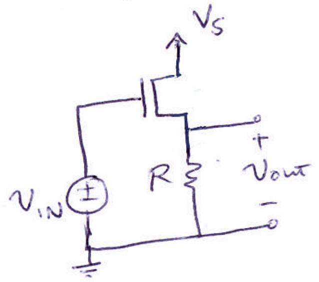
Problem 5.3



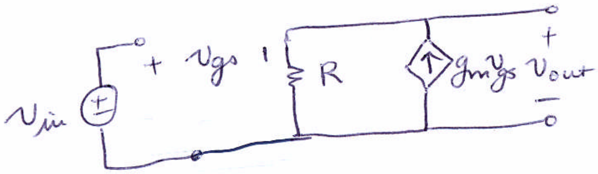
(A) From E5.1,

$$v_{OUT} = (v_{IN} - V_T) + \frac{1}{kR} - \sqrt{\frac{2(v_{IN} - V_T)}{kR} + \frac{1}{(kR)^2}}$$

$$\left.\frac{dv_{OUT}}{dv_{IN}}\right|_{v_{IN}=V_{IN}} = 1 - \frac{1}{\cancel{2}}\left(\frac{2(v_{IN} - V_T)}{KR} + \frac{1}{(KR)^2}\right)^{-1/2}\left(\frac{\cancel{2}}{KR}\right)\Bigg|_{v_{IN}=V_{IN}}$$

$$\boxed{\frac{v_{out}}{v_{in}} = 1 - \frac{1}{KR\sqrt{\frac{1}{(KR)^2} + \frac{2(V_{IN} - V_T)}{KR}}}}$$

(B)



$v_{gs} = v_{in} - v_{out}$

$g_m = K(V_{GS} - V_T)$

From above,

$$V_{GS} = V_{IN} - V_{OUT}$$
$$= V_{IN} - \left[(V_{IN} - V_T) + \frac{1}{kR} - \sqrt{\frac{2(V_{IN} - V_T)}{kR} + \frac{1}{(kR)^2}}\right]$$
$$V_{GS} = V_T - \frac{1}{kR} + \sqrt{\frac{2(V_{IN} - V_T)}{kR} + \frac{1}{(kR)^2}}$$

(C) From above, $v_{out} = R g_m v_{gs}$

$$= RK\left(\cancel{V_T} - \frac{1}{kR} + \sqrt{\frac{2(V_{IN} - V_T)}{kR} + \frac{1}{(kR)^2}} - \cancel{V_T}\right)(v_{in} - v_{out})$$

$$v_{out}\left(1 - 1 + RK\sqrt{\frac{2(V_{IN} - V_T)}{kR} + \frac{1}{(kR)^2}}\right) = \left(-1 + RK\sqrt{\frac{2(V_{IN} - V_T)}{kR} + \frac{1}{(kR)^2}}\right) v_{in}$$

$$\boxed{\frac{v_{out}}{v_{in}} = 1 - \frac{1}{RK\sqrt{\frac{2(V_{IN} - V_T)}{kR} + \frac{1}{(kR)^2}}}}$$

same as that found in part (B).

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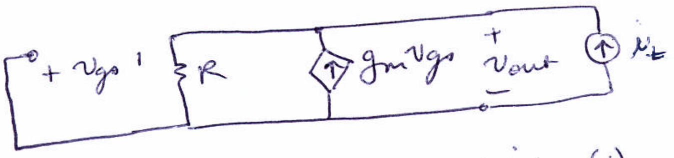
6.002 PSet 5 Sol'n

Problem 5.3 (D)

To find the Thevenin resistance, we set all independent sources to zero, making v_{in} a short circuit.



Node @ v_{out} : $\dfrac{v_{out}}{R} = g_m v_{gs} + i_t$ (1)

$$v_{gs} = 0 - v_{out} = -v_{out}$$

Substitute into (1), $v_{out}\left(\frac{1}{R} + g_m\right) = i_t$

$$\boxed{\frac{v_{out}}{i_t} = \frac{R}{1+Rg_m} = R_{Th}}$$

From part (C), a simpler expression for the small-signal gain can be found from

$v_{out} = g_m R\, v_{gs}$ (1)

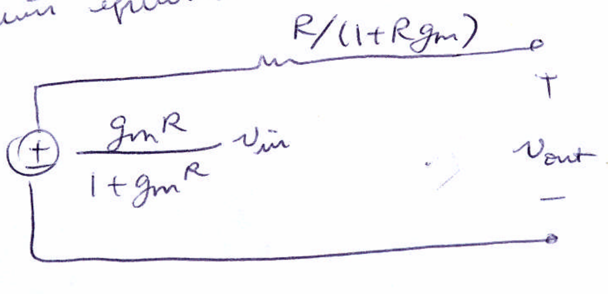
$v_{gs} = v_{in} - v_{out} \Rightarrow$ substitute into (1)

$v_{out}(1+g_mR) = g_m R\, v_{in} \Rightarrow \boxed{\dfrac{v_{out}}{v_{in}} = \dfrac{g_m R}{1+g_m R} \simeq 1}$

assuming $g_m R \gg 1$.

This derivation for the small-signal gain explains the name "source-follower" for this amplifier configuration i.e. the source follows the gate.

The Thevenin equivalent small-signal circuit B.



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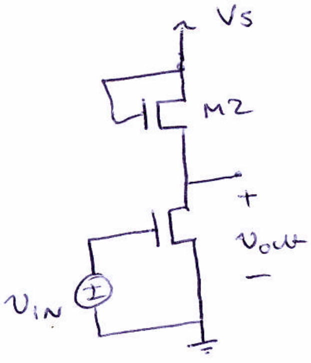
Problem 5.4

(A)

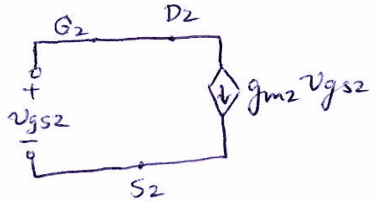


From E5.2, $v_{out} = -\sqrt{\frac{K_1}{K_2}}\,(v_{IN} - V_{T1}) + V_S - V_{T2}$

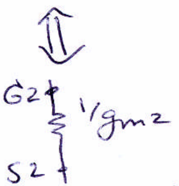
$$\boxed{\frac{v_{out}}{v_{in}} = \left.\frac{dv_{out}}{dv_{IN}}\right|_{v_{IN}=V_{IN}} = -\sqrt{\frac{K_1}{K_2}}}$$

small-signal voltage gain.

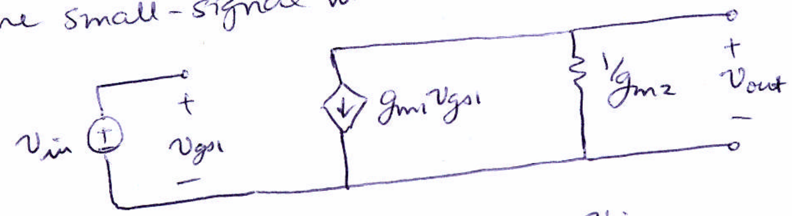
(B) M2:



Since M2 is connected so that the current through GS₂ is directly proportional to the voltage across these terminals, it looks like a resistor $1/g_{m2}$.



∴ The small-signal model of the amplifier reduces to



$g_{m1} = K_1(V_{IN} - V_{T1})$

$g_{m2} = K_2(V_S - V_{OUT} - V_{T2})$

(C) $v_{out} = -g_{m1}\left(\frac{1}{g_{m2}}\right)v_{gs1} \Rightarrow v_{gs1} = v_{in}$

$$\therefore \frac{v_{out}}{v_{in}} = \frac{-g_{m1}}{g_{m2}} = -\frac{K_1(V_{IN} - V_{T1})}{K_2(V_S - V_{OUT} - V_{T2})}$$

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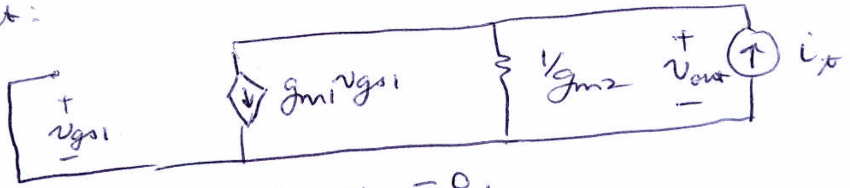
Problem 5.4 cont'd.

(C) Substituting V_{OUT} from E5.2, we have

$$\frac{v_{out}}{v_{in}} = \frac{-k_1 \; (V_{IN} - V_{T1})}{k_2\left[V_S - \left(-\sqrt{\frac{k_1}{k_2}}\,(V_{IN}-V_{T1}) + V_S - V_{T2}\right) - V_{T2}\right]}$$

$$\boxed{\frac{v_{out}}{v_{in}} = -\sqrt{\frac{k_1}{k_2}}}$$ as found in part (A).

(D) To find Thevenin resistance, we set the independent input source to zero and apply a test source at the output:



Since $v_{gs1} = 0$, $g_{m1}v_{gs1} = 0$.

$$\therefore R_{Th} = \frac{v_{out}}{i_t} = \frac{1}{g_{m2}}$$

The Thevenin small-signal circuit for our amplifier is