Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.002 – Circuits & Electronics
Spring 2008

Problem Set #8

Issued 4/2/08 – Due 4/9/08

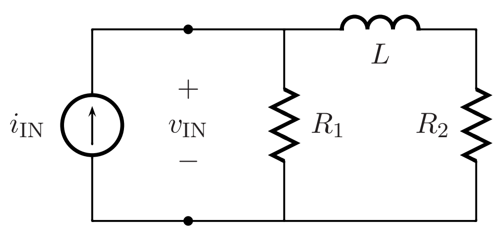
Exercise 8.1 (1 Point): A network comprising two resistors and one inductor is connected to a current source as shown below. When the source produces the current step

$$i_{IN}(t) = 1 \text{ mA } u(t) \quad ,$$

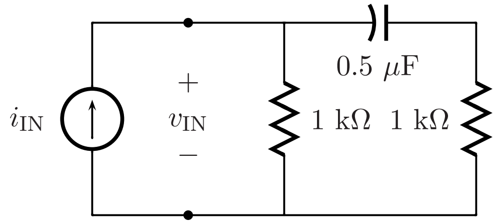
the voltage across the network is observed to be

$$v_{IN}(t) = 1 \text{ V } (1 + e^{-t/(1\ \mu\text{s})})\ u(t) \quad .$$

Given this information, determine R_1, R_2 and L. Try to do so without constructing and solving a differential equation. Rather, use the values of $i_{IN}(t > 0)$, $v_{IN}(0^+)$ and $v_{IN}(\infty)$, and the time constant with which v_{IN} evolves from $v_{IN}(0^+)$ to $v_{IN}(\infty)$, to complete this exercise.



Exercise 8.2: Determine $v_{IN}(t)$ in the network shown below when $i_{IN}(t)$ is the same 1-mA step studied in Exercise 8.1. Again, try to complete this exercise without constructing and solving a differential equation.



Problem 8.1 (2 Points): The network shown below includes two switches: S1 and S2. Prior to $t = 0$, both switches are closed, and the capacitor voltage $v(t)$ and inductor current $i(t)$ are both zero.

(A) At $t = 0$, S1 opens, and it remains open until $t = T_1$. Determine $v(t)$ and $i(t)$ for $0 \leq t \leq T_1$.

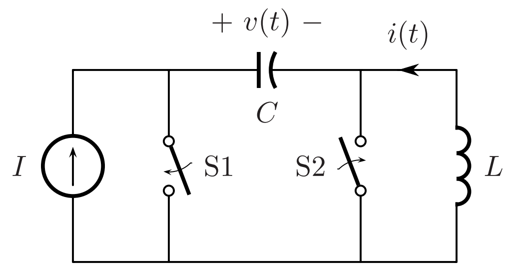
(B) At $t = T_1$, S1 closes as S2 simultaneously opens; the two switches change states so that $v(t)$ and $i(t)$ are continuous at $t = T_1$. The switches remain in their states until $v(t)$ goes to zero, at which time S2 closes. Define the time at which $v(t)$ goes to zero as $t = T_2$. Determine T_2, as well as $v(t)$ and $i(t)$ for $T_1 \leq t \leq T_2$.

(C) Both switches remain closed until $t = T_3$. Determine $v(t)$ and $i(t)$ for $T_2 \leq t \leq T_3$.

(D) At $t = T_3$, S1 again opens, and it remains open until $t = T_4$. Determine $v(t)$ and $i(t)$ for $T_3 \leq t \leq T_4$.

(E) Finally, at $t = T_4$, S1 closes as S2 again simultaneously opens; the two switches again change states so that $v(t)$ and $i(t)$ are continuous at $t = T_4$. The two switches remain in their states until $v(t)$ again goes to zero, at which time S2 closes. Define the time at which $v(t)$ again goes to zero as T_5. Determine T_5, as well as $v(t)$ and $i(t)$ for $T_4 \leq t \leq T_5$.

(F) Sketch and clearly label $v(t)$ and $i(t)$ for $0 \leq t \leq T_5$.



Problem 8.2 (2 Points): This problem is a continuation of Problem 8.1. It explores the use of energy conservation to analyze the operation of the network described therein.

(A) Determine the energy stored in the capacitor at $t = T_1$.

(B) The energy stored in the capacitor at $t = T_1$ is fully transferred to the inductor at $t = T_2$. Use this fact to determine $i(T_2)$. This answer should match your answer to Part B of Problem 8.1 when the latter is evaluated at $t = T_2$.

(C) Determine the energy stored in the capacitor at $t = T_4$.

(D) Use energy conservation to determine the energy stored in the inductor at $t = T_5$, and then determine $i(T_5)$. This answer should match the answer to Part E of Problem 8.1 when the latter is evaluated at $t = T_5$.

(E) Now let the switches move repetitively through the three-step cycle described in Problem 8.1: S1 initially open with S2 closed, next S1 closed with S2 open, finally both S1 and S2 closed. Assume that in each cycle S1 remains open for the duration T. Further, assume that S2 always closes when $v(t)$ reaches zero. Assuming that $v(t)$ and $i(t)$ are initially zero, determine $i(t)$ at the end of the nth switching cycle in terms of n, C, L, T and I.

Problem 8.3 (2 Points): iLab problem.

Problem 8.3: This problem studies Network A shown below to determine the current $i(t)$ that results when $v_{\rm IN}(t)$ is first a step, and second an impulse. It also illustrates that there is more than one method to determine the input-output response; you should think about which method you find easiest. Throughout this problem assume that both the inductor and capacitor are at rest prior to $t = 0$.

(A) Using the node method, derive a pair of coupled differential equations for the two unknown node voltages $e(t)$ and $v_{\rm C}(t)$. (You must differentiate once the equation that results from the application of KCL at the node at which $e(t)$ is defined in order to reduce the integral that comes from the constitutive law for the inductor.) Next, combine the two differential equations to form a single second-order differential equation for $v_{\rm C}(t)$. Finally, differentiate once the resulting differential equation, and substitute the constitutive law for the capacitor to form a second-order differential equation for $i(t)$ driven by $dv_{\rm IN}(t)/dt$.

(B) Since Network A has a single loop that carries the current $i(t)$, KVL can be conveniently used to determine a differential equation for $i(t)$. Apply KVL to Network A. Next, substitute the constitutive laws for the inductor, resistor and capacitor to form an equation that relates $i(t)$ to $v_{\rm IN}(t)$. Differentiate the equation once to reduce the integral that comes from the constitutive law for the capacitor. The resulting second-order differential equation should be the same as that found in Part A.

(C) In Network B, the inductor is replaced by a current source that represents its state variable, and the capacitor is replaced by a voltage source that represents its state variable. Using Network B, determine $v_{\rm L}(t)$ and $i(t)$ in terms of the input $v_{\rm IN}(t)$, and the two state variables $v_{\rm C}(t)$ and $i_{\rm L}(t)$. Next, recognize that

$$\frac{di_{\rm L}(t)}{dt} = \frac{1}{L}v_{\rm L}(t) \quad \& \quad \frac{dv_{\rm C}(t)}{dt} = \frac{1}{C}i(t) \quad ,$$

and combine these equations with those resulting from the analysis of Network B to form two coupled first-order differential equations for the states $i_{\rm L}(t)$ and $v_{\rm C}(t)$. Finally, combine the two first-order differential equations to determine a second-order differential equation for $i(t)$ driven by $dv_{\rm IN}(t)/dt$. The resulting second-order differential equation should be the same as those found in Parts A and B.

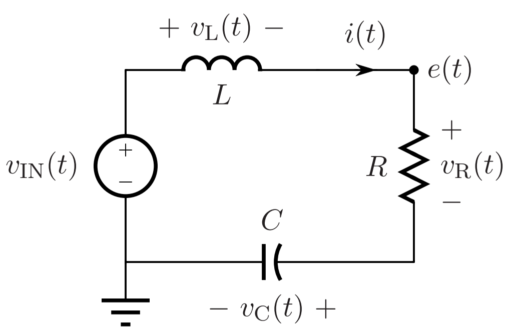
(D) Assume that $v_{\rm IN}(t)$ is a step input such that $v_{\rm IN}(t) = Vu_{-1}(t)$. For this input, determine $v_{\rm C}$, $v_{\rm L}$, $v_{\rm R}$, e, i and di/dt just after the step at $t = 0^+$. These initial conditions could be used to solve the differential equations found above.

(E) Rather than solve the second-order differential equation (or any of the equivalent coupled differential equations) for $i(t)$ directly, argue that $i(\infty) = 0$ so that $i(t)$ has no constant component. Further argue that $i(t)$ takes the form $i(t) = I\sin(\omega t + \phi)e^{-\alpha t}$ for $t \geq 0$. Determine I, ω, ϕ and α. Hint: first find ω and α from the differential equation, and then find I and ϕ from the initial conditions. Alternatively, solve for $i(t)$ by any method you wish.

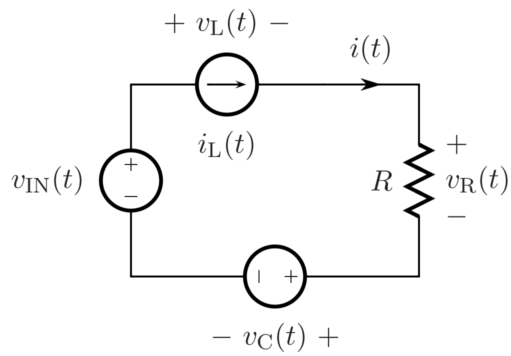
(F) Suppose that the input is a voltage impulse with area Λ where $\Lambda = \tau V$, V is the amplitude of the voltage step described above, and τ is a given time constant. Find the response of Network

A to the impulse. Hint: before solving this problem directly, consider the relation between step and impulse responses.

Save a copy of your answers to this problem. They will be useful during the pre-lab exercises for Lab #3.



Network A



Network B

Problem 8.4 (2 Points): This problem begins to develop a state-space analysis of the network shown below. However, it stops short of actually solving the associated differential equations. Rather, the focus here is on setting up those differential equations. In particular, the objective of this problem is to formulate a coupled set of three first-order differential equations that describe the evolution of the three network states i_{L}, v_{C1} and v_{C2}.

(A) In order to indicate knowledge of the network states, redraw the network with a current source of value i_{L} replacing the inductor, and voltage sources of values v_{C1} and v_{C2} replacing the two capacitors. Take care to retain the polarities of the states as shown in the network below. In addition, define v_{L}, i_{C1} and i_{C2} with consistent polarities.

(B) Using the redrawn circuit from Part A, determine v_{L}, i_{C1} and i_{C2} in terms of the network states i_{L}, v_{C1} and v_{C2}, and the independent sources I and V. Here, one approach is to use the node method. However, clever use of KCL, KVL and the constituitive laws for the resistors might be much quicker.

(C) Combine the results from Part B and the constitutive laws for the inductor and the two capacitors to develop three first-order differential equations that describe the evolution of i_{L}, v_{C1} and v_{C2}. Summarize the three differential equations in the form

$$\frac{d}{dt}\begin{bmatrix} i_{\mathrm{L}}(t) \\ v_{\mathrm{C1}}(t) \\ v_{\mathrm{C2}}(t) \end{bmatrix} = A \begin{bmatrix} i_{\mathrm{L}}(t) \\ v_{\mathrm{C1}}(t) \\ v_{\mathrm{C2}}(t) \end{bmatrix} + B \begin{bmatrix} I(t) \\ V(t) \end{bmatrix}$$

by finding the 3×3 matrix A and the 3×2 matrix B.