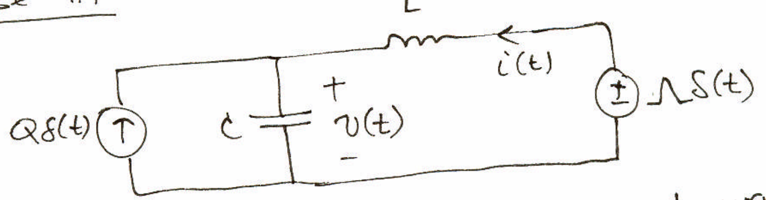
6.002 Problem Set #9 – Solutions

Exercise 9.1



Since the current through L does not want to change instantaneously, the impulse $Q\delta(t)$ in current will appear across the capacitor as

$$v(0^+) = \frac{1}{C}\int_{0^-}^{0^+} Q\delta(t)\,dt + V = \frac{Q}{C} + V.$$

Since the voltage across the capacitor does not want to change instantaneously, the impulse $\Lambda\delta(t)$ in voltage will appear across the inductor

$$i(0^+) = \frac{1}{L}\int_{0^-}^{0^+} \Lambda\delta(t)\,dt + I = \frac{\Lambda}{L} + I.$$

The system will oscillate at $\omega_0 = \frac{1}{\sqrt{LC}}$ rad/s.

$$v_1(t) = \left(\frac{Q}{C} + V\right)\cos\left(\frac{t}{\sqrt{LC}}\right)$$

$$i_1(t) = C\frac{dv_1}{dt} = -C\left(\frac{Q}{C} + V\right)\frac{1}{\sqrt{LC}}\sin\left(\frac{t}{\sqrt{LC}}\right)$$

v_1 and i_1 are the responses due to $Q\delta(t)$ and $v(0) = V$.

$$i_2(t) = \left(\frac{\Lambda}{L} + I\right)\cos\left(\frac{t}{\sqrt{LC}}\right)$$

$$v_2(t) = -L\frac{di_2}{dt} = -L\left(\frac{\Lambda}{L} + I\right)\left(\frac{-1}{\sqrt{LC}}\right)\sin\left(\frac{t}{\sqrt{LC}}\right).$$

$$\boxed{\begin{aligned} v(t) &= v_1(t) + v_2(t) = \left(\frac{Q}{C} + V\right)\cos\left(\frac{t}{\sqrt{LC}}\right) + \sqrt{\frac{L}{C}}\left(\frac{\Lambda}{L} + I\right)\sin\left(\frac{t}{\sqrt{LC}}\right) \\ i(t) &= i_1(t) + i_2(t) = \left(\frac{\Lambda}{L} + I\right)\cos\left(\frac{t}{\sqrt{LC}}\right) - \sqrt{\frac{C}{L}}\left(\frac{Q}{C} + V\right)\sin\left(\frac{t}{\sqrt{LC}}\right) \end{aligned}}$$

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6.002 Pset 9 - Sol'ns

Exercise 9.2

$R = 600\Omega$

From KCL: $C\frac{dv}{dt} + \frac{v}{R} + \frac{1}{L}\int v\,dt = 0$

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

$\omega_o = \frac{1}{\sqrt{LC}}$, $2\alpha = \frac{1}{RC} \Rightarrow \alpha = \frac{1}{2RC}$

$\omega_d = \sqrt{\omega_o^2 - \alpha^2} = \sqrt{\frac{1}{LC} - \frac{1}{4(RC)^2}}$

From the given response (see Appendix I), we measure

$f_d = \frac{1}{0.62ms} = 1.61\ kHz \Rightarrow \omega_d = 2\pi f_d \quad \therefore \omega_d = 10.13\ kHz$

$\frac{1}{\alpha} = 1.2ms$ $\qquad 1.2\ ms = 2RC \Rightarrow \boxed{C = 1\mu F}$

$10.13k = \sqrt{\frac{1}{LC} - \frac{1}{4(RC)^2}}$

$10.13K = \sqrt{\frac{1}{L(1\mu)} - \frac{1}{(1.2m)^2}}$

$\frac{1}{L(1\mu)} = 102.6M + 694.4k \Rightarrow \boxed{L = 9.7\ mH}$

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6.002 Pset 9 – Sol'ns

Problem 9.1

(A) 1) $v_{IN} = \mathrm{Im}\{V_{in}\, e^{j\omega t}\}$ $\qquad v_{OUT} = \mathrm{Im}\{\hat{V}_{out}\, e^{j\omega t}\}$

2)
$$v_{IN} = L\frac{di}{dt} + iR + v_{OUT}$$

$$i = \frac{v_{OUT}}{R_M} = \frac{v_{OUT}}{100}$$

$$\frac{L}{R_M}\frac{dv_{OUT}}{dt} + \frac{R}{R_M} v_{OUT} + v_{OUT} = v_{IN}$$

$$\boxed{\frac{dv_{OUT}}{dt} + \frac{R+R_M}{L} v_{OUT} = \frac{R_M}{L} v_{IN}}$$

3)
$$j\omega \hat{V}_{out} \cancel{e^{j\omega t}} + \frac{R+R_M}{L}\hat{V}_{out}\cancel{e^{j\omega t}} = \frac{R_M}{L} V_{in}\cancel{e^{j\omega t}}$$

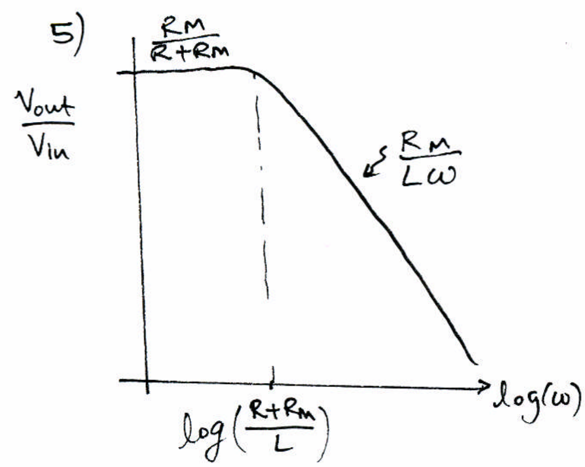
$$\hat{V}_{out} = \frac{(R_M/L)\, V_{in}}{\frac{R+R_M}{L} + j\omega}$$

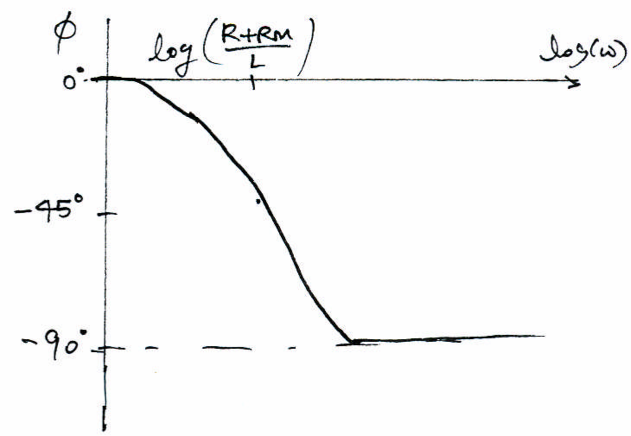
$$\boxed{\hat{V}_{out} = \frac{R_M/(R+R_M)\; V_{in}}{1 + \frac{L}{R+R_M} j\omega}}$$

4)
$$\boxed{V_{out} = |\hat{V}_{out}| = \frac{R_M}{R+R_M}\cdot\frac{V_{in}}{\sqrt{1+\omega^2\left(\frac{L}{R+R_M}\right)^2}}}$$

$$\boxed{\phi = -\tan^{-1}\left(\frac{L\omega}{R+R_M}\right)}$$

5)





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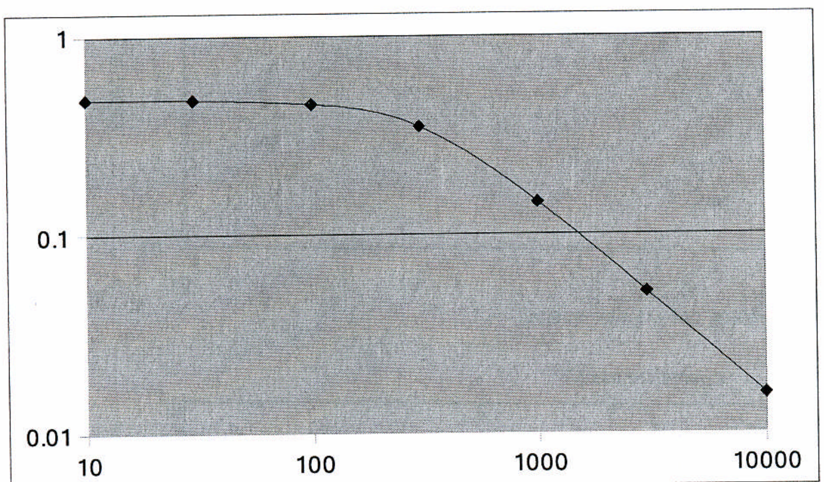
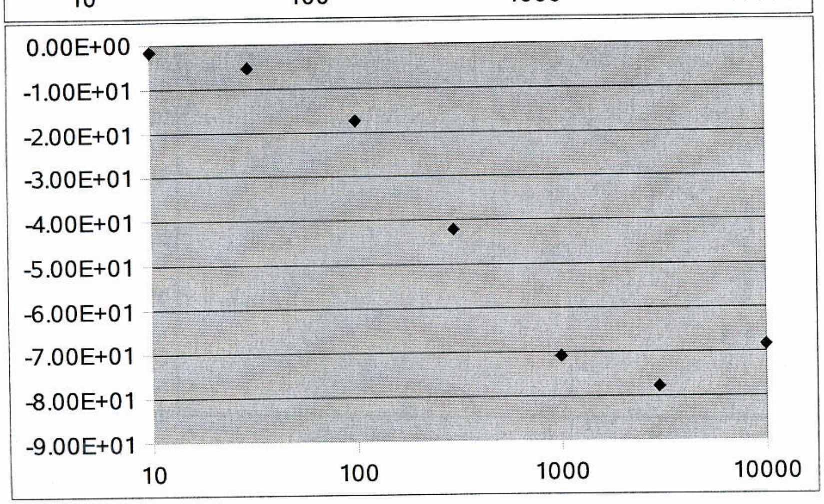
6.002 Problem Set 9 – Sol'ns

Problem 9.1 contd.

(B) See below for plotted data. Please note that measurements could only be made up to f = 10kHz because of the maximum sampling rate on the oscilloscope.

f (Hz)	vi (V)	vout (V)	vout/vin	phi (s)	theta
10	0.79	0.38	0.48	4.40E-04	-1.58E+00
30	0.8	0.38	0.48	4.90E-04	-5.29E+00
100	0.8	0.37	0.46	4.80E-04	-1.73E+01
300	0.86	0.3	0.35	3.90E-04	-4.21E+01
1000	0.96	0.14	0.15	1.97E-04	-7.09E+01
3000	1	0.05	0.05	7.20E-05	-7.78E+01
10000	0.96	0.02	0.02	1.90E-05	-6.84E+01





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6.002 PSet #9 - Sol'ns

Problem 9.1 cont'd.

(c) At low f, the inductor acts as a short circuit, so the gain is

$$\frac{v_{OUT}}{v_{IN}} = \frac{R_M}{R + R_M} = \frac{100}{R + 100} = \frac{0.381\ V}{0.79\ V}$$

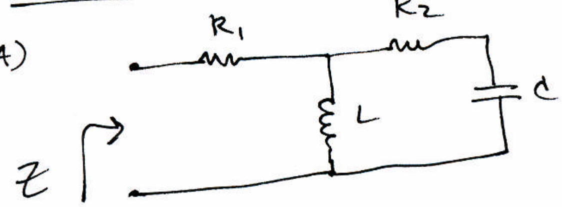
$$79 = 0.381R + 38.1 \Rightarrow \boxed{R = 107\ \Omega}$$

From P9.1(A), the corner frequency of the circuit is

$$\omega_c = \frac{R + R_M}{L} = \frac{100 + 107}{L} = 1.885\ \text{krad/s} \Rightarrow \boxed{L = 0.1\ H}$$

Problem 9.2

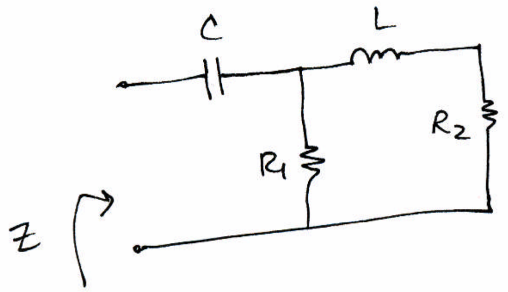
(A)



$$Z = \left(R_2 + \frac{1}{j\omega C}\right) \| j\omega L + R_1$$

$$= \frac{(j\omega R_2 C + 1)/j\omega C \cdot j\omega L}{j\omega L + \frac{j\omega R_2 C + 1}{j\omega C}} + R_1$$

$$\boxed{Z = \frac{j\omega L(j\omega R_2 C + 1)}{j\omega R_2 C + 1 - \omega^2 LC} + R_1}$$



$$Z = (j\omega L + R_2) \| R_1 + \frac{1}{j\omega C}$$

$$\boxed{Z = \frac{(j\omega L + R_2) R_1}{j\omega L + R_2 + R_1} + \frac{1}{j\omega C}}$$

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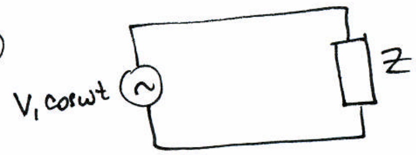
6.002 PSet #9 - Sol'ns

Problem 9.2 cont'd.

(B)



In both cases,

$$\hat{I} = \frac{V_1}{Z}$$

In the upper circuit,

$$\hat{I} = \frac{V_1}{\dfrac{j\omega(j\omega R_2C+1)}{(j\omega R_2C+1-\omega^2LC} + R_1}$$

$$= \frac{V_1(1-\omega^2LC + j\omega R_2C)}{R_1 - \omega^2LCR_1 - \omega^2R_2C + j\omega + j\omega R_1R_2C}$$

$$I_1\cos(\omega t+\phi) = |\hat{I}|\cos(\omega t + \angle\hat{I})$$

$$\boxed{\begin{aligned} \therefore I_1 &= \frac{V_1\sqrt{(1-\omega^2LC)^2 + \omega^2R_2^2C^2}}{\sqrt{(R_1-\omega^2LCR_1-\omega^2R_2C)^2 + (\omega+\omega R_1R_2C)^2}} \\ \phi &= \tan^{-1}\left(\frac{\omega R_2C}{1-\omega^2LC}\right) - \tan^{-1}\left(\frac{\omega+\omega R_1R_2C}{R_1-\omega^2LCR_1-\omega^2R_2C}\right) \end{aligned}}$$

For the lower circuit,

$$\hat{I} = \frac{V_1}{\dfrac{j\omega C(j\omega LR_1 + R_2R_1) + j\omega L + R_1 + R_2}{(j\omega L+R_1+R_2)\,j\omega C}}$$

$$= \frac{V_1(-\omega^2LC + j\omega C(R_1+R_2))}{R_1+R_2-\omega^2LCR_1 + j\omega(L+CR_1R_2)}$$

$$I_1\cos(\omega t+\phi) = |\hat{I}|\cos(\omega t + \angle\hat{I})$$

$$\boxed{\begin{aligned} \therefore I_1 &= \frac{V_1\omega C\sqrt{\omega^2L^2 + (R_1+R_2)^2}}{\sqrt{(R_1+R_2-\omega^2LCR_1)^2 + (\omega L + CR_1R_2\omega)^2}} \\ \phi &= \frac{\pi}{2} + \tan^{-1}\left(\frac{\omega L}{R_1+R_2}\right) - \tan^{-1}\left(\frac{\omega(L+CR_1R_2)}{R_1+R_2-\omega^2LCR_1}\right) \end{aligned}}$$

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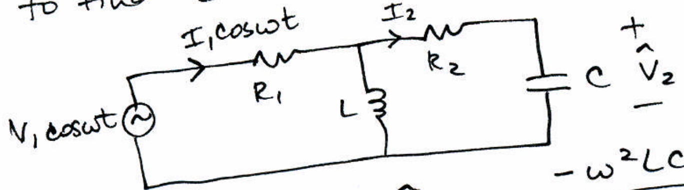
6.002 Pset #9 - Sol'n's

Problem 9.2 cont'd.

(C) In part (B), we calculated $I_1 \cos(\omega t + \phi_1)$, the current in Port #1. Now, we can use a current divider expression to find V_2 and ϕ_2.



$$\hat{I}_2 = \frac{j\omega L}{j\omega L + R_2 + 1/j\omega C}\hat{I}_1 = \frac{-\omega^2 LC}{1-\omega^2 LC + j\omega R_2 C}\hat{I}_1$$

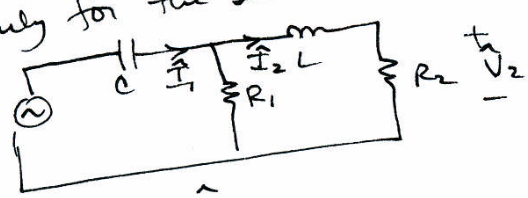
$$\hat{V}_2 = \frac{1}{j\omega C}\cdot\hat{I}_2 = \frac{j\omega L\,\hat{I}_1}{1-\omega^2 LC + j\omega R_2 C}$$

$$V_2\cos(\omega t + \phi_2) = |\hat{V}_2|\cos(\omega t + \angle\hat{V}_2)$$

$$\boxed{|\hat{V}_2| = \frac{\omega L}{\sqrt{(1-\omega^2 LC)^2 + (\omega^2 R_2^2 C^2)}}|I_1| = \frac{\omega L V_1}{\sqrt{(R_1 - \omega^2 LCR_1 - \omega^2 R_2 C)^2 + (\omega + \omega R_1 R_2 C)^2}}}$$

$$\boxed{\phi_2 = \frac{\pi}{2} - \tan^{-1}\left(\frac{\omega R_2 C}{1-\omega^2 LC}\right) + \angle I_1 = \frac{\pi}{2} - \tan^{-1}\left(\frac{\omega + \omega R_1 R_2 C}{R_1 - \omega^2 LCR_1 - \omega^2 R_2 C}\right)}$$

Similarly for the second circuit



$$\hat{I}_2 = \frac{R_1}{R_1 + R_2 + j\omega L}\hat{I}_1$$

$$\hat{V}_2 = R_2\hat{I}_2 = \frac{R_1 R_2}{R_1 + R_2 + j\omega L}\hat{I}_1$$

$$\boxed{|\hat{V}_2| = \frac{R_1 R_2}{\sqrt{(R_1+R_2)^2 + (\omega L)^2}}|\hat{I}_1| = \frac{R_1 R_2 \omega C V_1}{\sqrt{(R_1 + R_2 - \omega^2 LCR_1)^2 + (\omega L + CR_1 R_2 \omega)^2}}}$$

$$\boxed{\phi_2 = \angle\hat{I}_1 - \tan^{-1}\left(\frac{\omega L}{R_1 + R_2}\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{\omega(L + CR_1 R_2)}{R_1 + R_2 - \omega^2 LCR_1}\right)}$$

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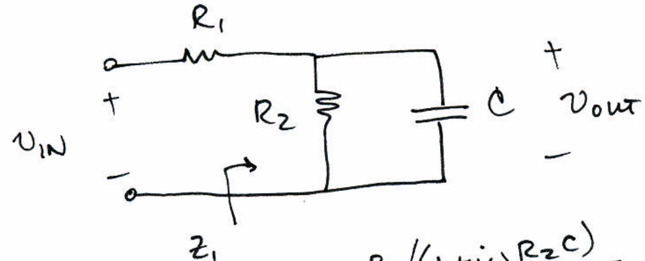
6.002 Pset #9 - Sol'us

Problem 9.3

The low-frequency gain of the network is 0.8, and it is configured in a low-pass configuration with corner frequency $f_c = 700$ Hz. These details tell us that the three elements are configured



$$Z_1 = \frac{R_2/j\omega C}{R_2 + \frac{1}{j\omega C}} = \frac{R_2}{1 + j\omega R_2 C}$$

$$\frac{v_{OUT}}{v_{IN}} = \frac{Z_1}{Z_1 + R_1} = \frac{R_2/(1+j\omega R_2 C)}{\frac{R_2}{1+j\omega R_2 C} + R_1} = \frac{R_2}{R_1 + R_2 + j\omega R_1 R_2 C}$$

At $\omega = 0$ $\quad \frac{v_{OUT}}{v_{IN}} = \frac{R_2}{R_1 + R_2} = 0.8 \Rightarrow 0.8 R_1 = 0.2 R_2$

$$R_2 = 4R_1$$

$$\omega_c = \frac{R_1 + R_2}{R_1 R_2 C} = \frac{R_1 + 4R_1}{4R_1^2 (0.1\mu)} = 800$$

$$5 = (0.32\,m) R_1 \Rightarrow \boxed{R_1 = 15.625\,K\Omega}$$

$$\boxed{R_2 = 62.5\,K\Omega}$$

(see Appendix II for values).

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6.002 PSet #9 – Sol'ns

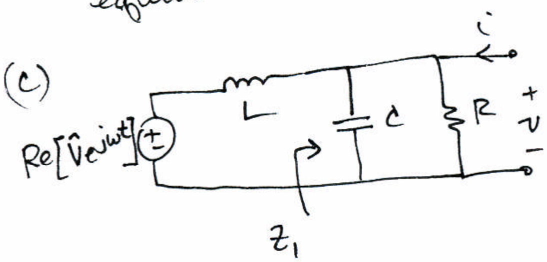
Problem 9.4

(A) Since the same laws apply as with purely resistive circuits, we get

$$\hat{V}_T = Z_N \hat{I}_N \qquad \hat{I}_N = \frac{\hat{V}_T}{Z_T} \qquad Z_T = Z_N$$

(B) We see that each network has a sinusoidal source at frequency ω rad/s, and the output will have a magnitude and phase-shifted version of the same frequency sinusoidal source. Since both networks are equivalent as long as they obey the relationship in (A), they are Thevenin and Norton equivalents.

(C)



$$\hat{V}_T = \frac{Z_1}{j\omega L + Z_1}\hat{V} \qquad Z_1 = \frac{R/j\omega C}{R + \frac{1}{j\omega C}} = \frac{R}{1 + j\omega CR}$$

$$\hat{V}_T = \frac{R/(1+j\omega CR)\,\hat{V}}{j\omega L + \frac{R}{(1+j\omega CR)}}$$

$$\boxed{\hat{V}_T = \frac{R\hat{V}}{R + j\omega L - \omega^2 LCR}}$$

$$Z_T = Z_1 \| j\omega L = \frac{Rj\omega L/(1+j\omega CR)}{j\omega L + \frac{R}{(1+j\omega CR)}} \Rightarrow \boxed{Z_T = \frac{j\omega LR}{R - \omega^2 LCR + j\omega L}}$$

(D) For network #4, $Z_T = R_T + \frac{1}{j\omega C_T} = R_T - \frac{j}{\omega C_T}$ (1)

From part (C), $Z_T = \dfrac{j\omega LR\,(R - \omega^2 LCR - j\omega L)}{(R - \omega^2 LCR)^2 + \omega^2 L^2}$

$$Z_T = \frac{\omega^2 L^2 R + j\omega LR^2(1-\omega^2 LC)}{(R-\omega^2 LCR)^2 + \omega^2 L^2} \quad (2)$$

Equating the real and imaginary parts of (1) and (2), we get

$$\boxed{R_T = \frac{\omega^2 L^2 R}{(R-\omega^2 LCR)^2 + \omega^2 L^2} \quad \text{and} \quad C_T = -\frac{\omega^2 L^2 + (R-\omega^2 LCR)^2}{\omega^2 L R^2 (1-\omega^2 LC)}}$$

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6.002 Problem Set 9 Sol'ns

Problem 9.4 cont'd.

(D) cont'd.

For network #5,

$$Z_T = R_T + j\omega L_T \quad (3)$$

Equating the real and imaginary parts of (2) and (3), we get

$$\boxed{R_T = \frac{\omega^2 L^2 R}{(R - \omega^2 LCR)^2 + \omega^2 L^2} \quad \text{and} \quad L_T = \frac{LR^2(1-\omega^2 LC)}{(R-\omega^2 LCR)^2 + \omega^2 L^2}}$$

Network #4 is preferred when we do not want low frequency signals to pass since the capacitor behaves as an open circuit at low frequencies. Similarly, Network #5 is preferred when we do not want high frequency signals to pass since the inductor behaves as an open circuit at high frequencies.

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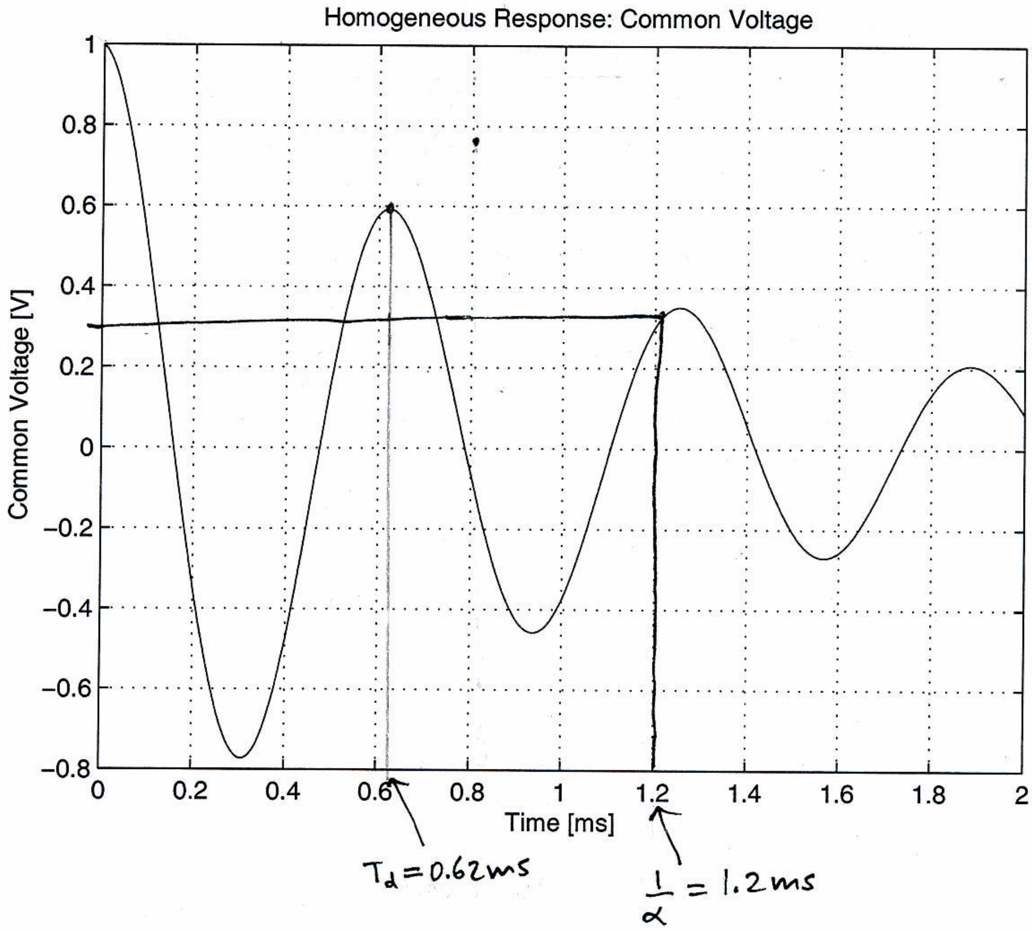
6.002 Pset #9 – Sol'ns

Appendix I

Exercise 9.2



Appendix II

Problem 9.3