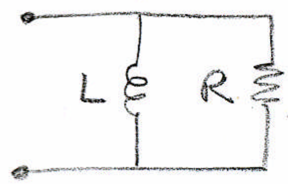
6.002 - Spring 2008 - PSET 10
Solutions - 4/23/08

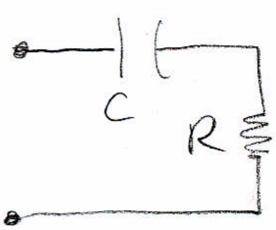
Exercise 10.1



Parallel impedances combine like parallel resistors: $Z_{tot} = \frac{1}{\frac{1}{j\omega L} + \frac{1}{R}} = \frac{j\omega RL}{R + j\omega L}$

For very low frequencies, $Z_{tot}(\omega \to 0) = 0$ as the inductor approaches a short circuit.

For very high frequencies, $Z_{tot}(\omega \to \infty) = R$, since $j\omega L >> R$ and the inductor approaches an open circuit.



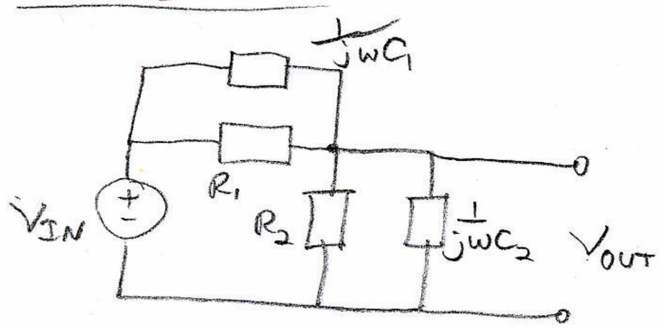
Series impedances combine like series resistors:

$$Z_{tot} = R + \frac{1}{j\omega C}$$

At low frequencies, the capacitor impedance dominates, approaching an open circuit and $Z_{tot}(\omega \to 0) \to \infty$

At high frequencies, the capacitor approaches a short circuit and $Z_{tot}(\omega \to \infty) = R$

Exercise 10.2



Combining // impedances:

$$\frac{1}{\frac{1}{R}+j\omega C} = \frac{R}{1+j\omega RC}$$

Using a voltage divider with the 2 parallel combinations:

$$V_{OUT} = V_{IN}\left(\frac{\frac{R_2}{1+j\omega R_2C_2}}{\frac{R_2}{1+j\omega R_2C_2}+\frac{R_1}{1+j\omega R_1C_1}}\right)$$

and multiplying through by both denominators gives

OR

$$V_{OUT} = V_{IN}\,\frac{R_2(1+j\omega R_1C_1)}{R_2(1+j\omega R_1C_1)+R_1(1+j\omega R_2C_2)}$$

If $\boxed{R_1C_1 = R_2C_2}$ then $(1+j\omega R_1C_1)=(1+j\omega R_2C_2)$ and

$$V_{OUT} = V_{IN}\left(\frac{R_2}{R_1+R_2}\right) \text{ and } \phi = 0$$

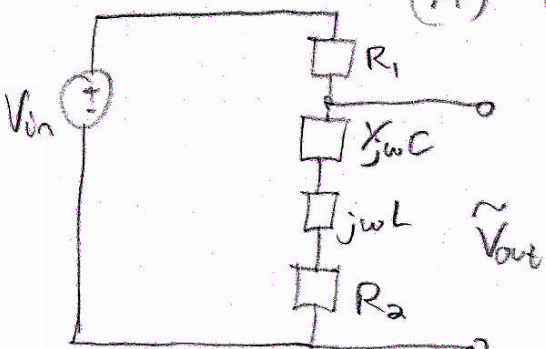
for $V_{out}=0.1V_{in}$, $\frac{R_2}{R_1+R_2}=\frac{1}{10}$ $\boxed{R_1 = 9R_2}$

using $R_1C_1=R_2C_2$ and $R_1=9R_2$, $9R_2C_1=R_2C_2$

$$\boxed{C_2 = 9C_1}$$

Problem 10.1

(A) Using a voltage divider



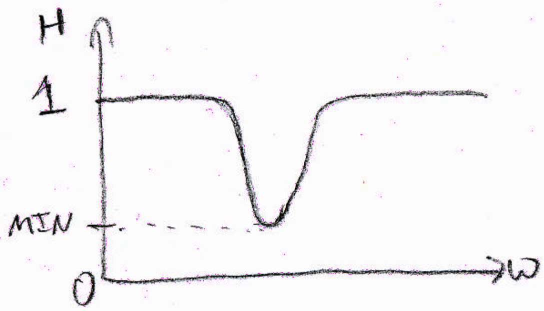
$$\tilde{V}_{out} = V_{in}\left(\frac{1/j\omega C + j\omega L + R_2}{(R_1+R_2) + 1/j\omega C + j\omega L}\right)$$

$$\tilde{V}_{out} = V_{in}\left(\frac{(1-\omega^2 LC) + j\omega C R_2}{(1-\omega^2 LC) + j\omega C(R_1+R_2)}\right)$$

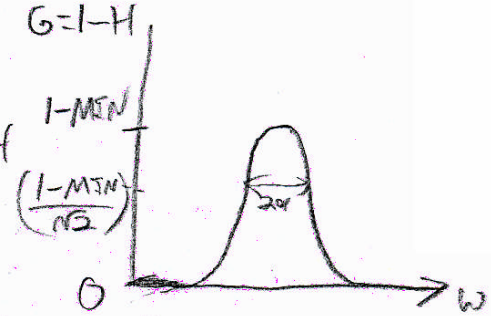
$$\boxed{V_{out} = V_{in}\sqrt{\frac{(1-\omega^2 LC)^2 + (\omega C R_2)^2}{(1-\omega^2 LC)^2 + (\omega C(R_1+R_2))^2}}}$$

$$\boxed{\phi = \tan^{-1}\left(\frac{\omega C R_2}{1-\omega^2 LC}\right) - \tan^{-1}\left(\frac{\omega C(R_1+R_2)}{1-\omega^2 LC}\right)}$$

(B) For the Noth filter, $H = \frac{\tilde{V}_{out}}{V_{in}}$ has the general shape:



but if we define the gain $G = 1-H$



Now $G = 1-H$ looks like a bandpass filter and we can use the bandpass definition of Q

Problem 10.1 (continued)

(B) so $G = 1 - H = 1 - \frac{\tilde{V}_{out}}{V_{in}} = 1 - \frac{(1-\omega^2 LC) + j\omega C R_2}{(1-\omega^2 LC) + j\omega C(R_1+R_2)}$

$$G = \frac{\cancel{(1-\omega^2 LC)} + j\omega C(R_1+\cancel{R_2}) - \cancel{(1-\omega^2 LC)} - \cancel{j\omega C R_2}}{(1-\omega^2 LC) + j\omega C(R_1+R_2)}$$

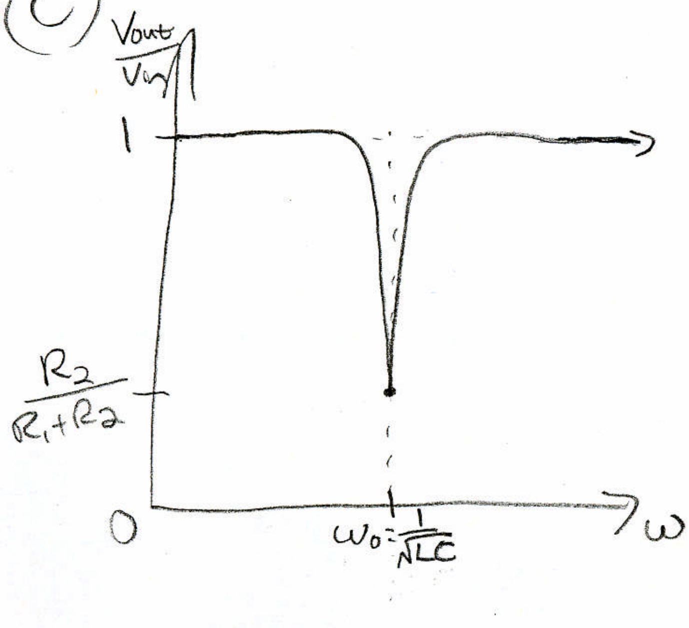
$$G = \frac{j\omega C R_1}{(1-\omega^2 LC) + j\omega C(R_1+R_2)}$$

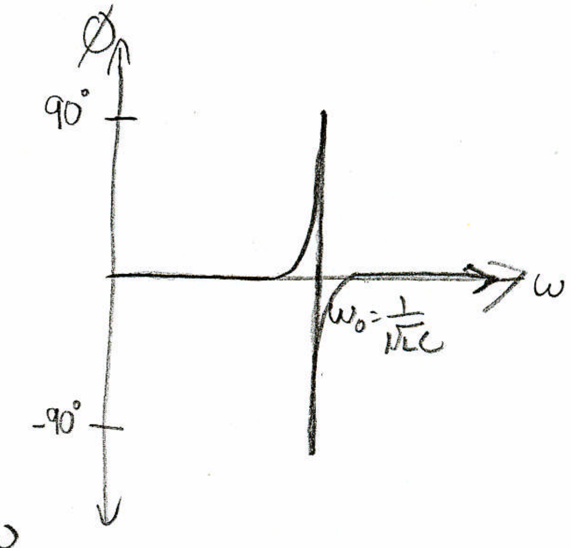
this is the same form as a band pass filter, and 2α is the coefficient of the 1st order, $j\omega$, term, after dividing through by LC to give 1 for the coefficient of the highest, 2nd order, ω^2 term.

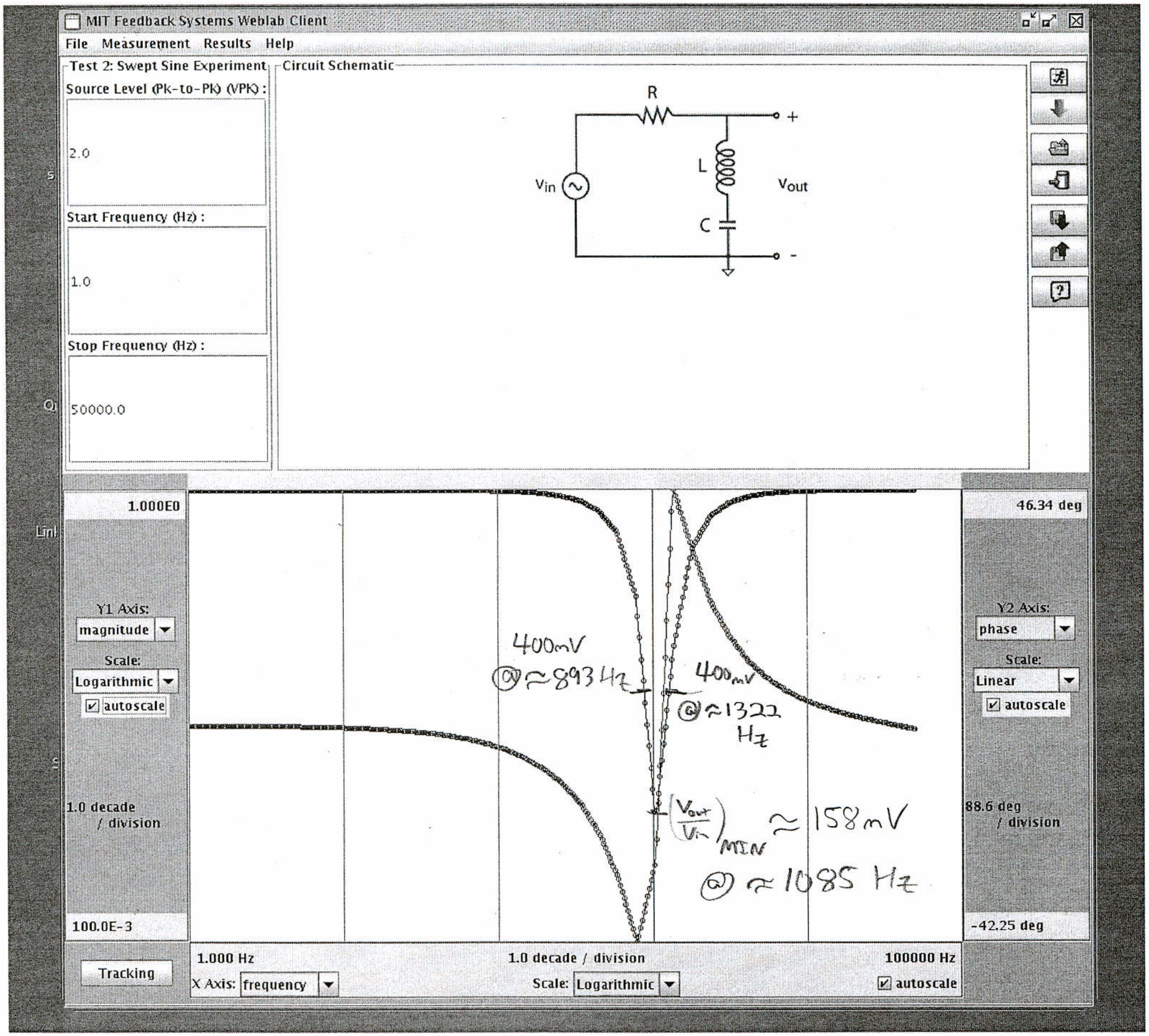
$2\alpha = \frac{R_1+R_2}{L}$ and $Q \equiv \frac{\omega_0}{2\alpha} = \frac{1}{\sqrt{LC}} \frac{L}{R_1+R_2}$

$$\boxed{Q = \sqrt{\frac{L}{C}} \cdot \frac{1}{R_1+R_2}}$$

(C)







MIT Feedback Systems Weblab Client
File Measurement Results Help
Test 2: Swept Sine Experiment
Source Level (Pk-to-Pk) (VPK):
2.0
Start Frequency (Hz):
1.0
Stop Frequency (Hz):
50000.0
Circuit Schematic
R
L
C
Vin
Vout
+
-
1.000E0
Y1 Axis:
magnitude
Scale:
Logarithmic
autoscale
1.0 decade / division
100.0E-3
Tracking
1.000 Hz
X Axis: frequency
1.0 decade / division
Scale: Logarithmic
100000 Hz
autoscale
46.34 deg
Y2 Axis:
phase
Scale:
Linear
autoscale
88.6 deg / division
-42.25 deg
400mV @≈893Hz
400mV @≈1322 Hz
(Vout/Vin)MIN ≈ 158mV @≈1085 Hz

Problem 10.1 (continued)

(D) See attached screen shot

(E) given $L=98\,mH$, the minimum output voltage occurs at the resonant frequency, $\omega_0 \approx 1085 \cdot 2\pi$

$$\omega_0 = \frac{1}{\sqrt{LC}} \Rightarrow C = \frac{1}{\omega_0^2 L} \approx \frac{1}{(2\pi \cdot 1085)^2 \cdot 98 \cdot 10^{-3}}$$

$$\boxed{C \approx 0.22\,\mu F}$$

at $\omega = \omega_0$, the minimum value $\left(\frac{V_{out}}{V_{in}}\right)_{min} = \frac{\omega_0 C R_2}{\omega_0 C (R_1 + R_2)} = \frac{R_2}{R_1 + R_2}$

From the plot, $\left(\frac{V_{out}}{V_{in}}\right)_{MIN} \approx 158\,mV$

Can also graphically find the bandwidth, 2α, as the difference in frequency when $G = \frac{G_{max}}{\sqrt{2}} = 1 - H$, $H = \frac{V_{out}}{V_{in}}$

$G_{max} = 1 - \left(\frac{V_{out}}{V_{in}}\right)_{MIN}$ so $1 - \frac{V_{out}}{V_{in}} = \frac{1 - \left(\frac{V_o}{V_{in}}\right)_{MIN}}{\sqrt{2}}$ with $\left(\frac{V_{out}}{V_{in}}\right)_{MIN} = 0.158V$

$\left(\frac{V_{out}}{V_{in}}\right) = 1 - \frac{1 - \left(\frac{V_{out}}{V_{in}}\right)_{MIN}}{\sqrt{2}} \approx 0.40V$ at $f \approx 1322$ and $893\,Hz$

$2\alpha \approx (1322 - 893) \cdot 2\pi$ = and from (B) $2\alpha = \frac{R_1 + R_2}{L}$

so (1) $\frac{R_2}{R_1 + R_2} = 0.158$ and (2) $2\alpha L = R_1 + R_2$

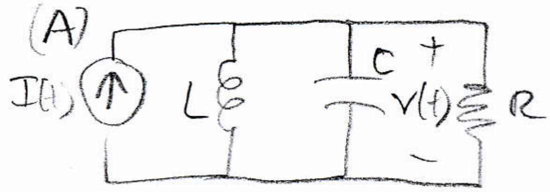
$$R_2 = 0.158(2\alpha L) \approx 0.158\,(1322-893) \cdot 2\pi\,(98 \cdot 10^{-3}) \approx 42\,\Omega$$

$$R_1 \approx 2\alpha L - R_2 \approx 222\,\Omega$$

$$\boxed{R_1 \approx 220\,\Omega,\ R_2 \approx 42\,\Omega}$$

Problem 10.2

(A)



The voltage $v(t)$ is equal to the input current times the parallel combination of the three impedances:

$$\tilde{V} = I \cdot \left(\frac{1}{\frac{1}{j\omega L} + j\omega C + \frac{1}{R}} \right) = I \left(\frac{j\omega LR}{R(1-\omega^2 LC) + j\omega L} \right)$$

$$\boxed{V(\omega) = |\tilde{V}| = I \cdot \frac{\omega LR}{\sqrt{R^2(1-\omega^2 LC)^2 + (\omega L)^2}}}$$

$$\boxed{\phi(\omega) = \frac{\pi}{2} - \tan^{-1}\left(\frac{\omega L}{R(1-\omega^2 LC)} \right)}$$

(B) To maximize V, minimize denominator with $\omega^2 LC = 1$, $\boxed{\omega = \frac{1}{\sqrt{LC}}}$

(C) $L = 365\mu H$, need

$$2\pi \cdot 540{,}000 \leq \omega = \frac{1}{\sqrt{LC}} \leq 2\pi \cdot 1{,}600{,}000$$

$$L \cdot (2\pi \cdot 540e3)^2 \leq \frac{1}{C} \leq L \cdot (2\pi \cdot 1600e3)^2$$

$$\frac{1}{365 \cdot 4\pi^2 \cdot 1600^2} \leq C \leq \frac{1}{365 \cdot 4\pi^2 \cdot 540^2}$$

$$\boxed{27.1 pF \leq C \leq 238 pF}$$

(D) $\frac{V(\omega)}{V(\omega_0)} = \frac{1}{4}$, where ω_0 is the center frequency $\omega_0 = 2\pi \cdot 1\,MHz$ and $V(\omega_0) = IR$

$L = 365\,\mu H$, $C = \frac{1}{\omega_0^2 L} \approx 69.4\,pF$

$$\frac{V(\omega)}{V(\omega_0)} = \frac{\dfrac{I\omega LR}{\sqrt{R^2(1-\omega^2LC)^2+(\omega L)^2}}}{IR} = \frac{1}{4}$$

$$\frac{1}{16} = \frac{\omega^2L^2}{R^2(1-\omega^2LC)^2+\omega^2L^2} \text{ OR } R^2(1-\omega^2LC)^2 = 15\omega^2L^2$$

$$R^2 = \frac{15\omega^2L^2}{(1-\omega^2LC)^2} \text{ and } R = \pm\frac{\omega L\sqrt{15}}{(1-\omega^2LC)}$$ with the sign chosen to give a positive R

using $L = 365\,\mu H$, $C = 69.4\,pF$ and $\omega = 2\pi \cdot 1.005\,MHz$

gives $\boxed{R \approx 888\,k\Omega}$ can check that $\omega = 2\pi \cdot .995\,MHz$ gives same

(E) $$\left.\frac{V(\omega)}{V(\omega_0)}\right|_{\substack{\omega = 1.010\,MHz \cdot 2\pi \\ \text{or } \omega = 0.990\,MHz \cdot 2\pi}} = \frac{\omega L}{\sqrt{R^2(1-\omega^2LC)^2+(\omega L)^2}} \boxed{\approx 0.13}$$

For Q - from (A) replace $j\omega \to s$ or $\frac{d}{dt}$ to find the Parallel RLC

$Q \equiv \frac{\omega}{2\alpha}$ characteristic equation: $\tilde{V}\left(s^2 + s\cdot\frac{1}{RC} + \frac{1}{LC}\right) = \frac{I \cdot s}{C}$

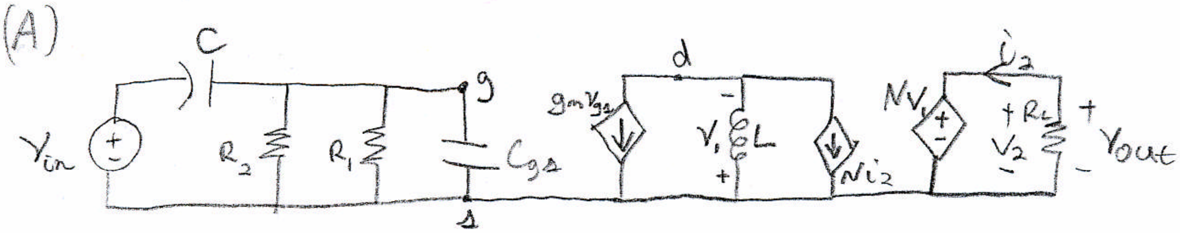
So $2\alpha = \frac{1}{RC}$ and $Q = 2\pi f \cdot RC \Rightarrow \boxed{Q \approx 387}$

Problem 10.2 (continued)

(F) As $v(t)$ decays proportionally to $e^{-\alpha t}$ after 3 time costants, $\tau = \frac{1}{\alpha} = 2RC \approx 123$ μsec, the voltage from the first station will have decayed to $\approx 5\%$ of the original value. Also, Q is the number of oscillations before the energy in the transient has decayed to practically zero, so the time is $Q \cdot T = \frac{Q}{f_{\text{first station}}} \approx \frac{Q}{1 \cdot 10^6}$ $\boxed{\approx 387 \text{ μsec.}}$

Problem 10.3

(A)



(B) $$V_{gs} = V_{in}\left(\frac{\frac{1}{j\omega C_{gs} + \frac{1}{R_1} + \frac{1}{R_2}}}{\frac{1}{j\omega C_{gs} + \frac{1}{R_1} + \frac{1}{R_2}} + \frac{1}{j\omega C}}\right) = \frac{j\omega C}{j\omega C + j\omega C_{gs} + \frac{1}{R_1} + \frac{1}{R_2}}$$

$$V_{gs} = V_{in} = \frac{j\omega C}{j\omega(C + C_{gs}) + \frac{R_1 + R_2}{R_1 R_2}} = V_{in}\frac{j\omega C R_{eq}}{j\omega R_{eq}(C + C_{gs}) + 1} \qquad R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$$

$$V_{out} = N \cdot V_1 \,, \quad V_1 = i_L \cdot j\omega L = (g_m V_{gs} + N i_2)\, j\omega L \quad i_2 = -\frac{V_{out}}{R_L}$$

$$\frac{V_{out}}{N} = \left(g_m V_{gs} - \frac{N V_{out}}{R_L}\right) j\omega L \;; \quad V_{out}\left(1 + \frac{N^2 j\omega L}{R_L}\right) = N j\omega L g_m V_{gs}$$

$$V_{out} = V_{gs}\left(\frac{j\omega L \cdot N \cdot g_m \cdot R_L}{R_L + jN^2\omega L}\right)$$ combine with V_{gs}

Problem 10.3 (continued)

(B) Combining $V_{gs} = f(V_{in})$ with $V_{out} = f(V_{gs})$ and (C)

$$V_{out} = V_{in}\left(\frac{j\omega C R_{eq}}{1 + j\omega R_{eq}(C + C_{gs})}\right)\left(\frac{j\omega L \cdot N \cdot g_m R_L}{R_L + jN^2\omega L}\right)$$

$$\boxed{G = \frac{V_{out}}{V_{in}} = \frac{-\omega^2 L C R_{eq} R_L N \cdot g_m}{\left(1 + j\omega R_{eq}(C + C_{gs})\right)\left(R_L + jN^2\omega L\right)}}$$

where $R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$

$G \to 0$ as $\omega \to 0$

$G \to \frac{-\omega^2 L C R_{eq} R_L N \cdot g_m}{-\omega^2 R_{eq} N^2 L (C + C_{gs})} = \frac{g_m R_L C}{N(C + C_{gs})}$ as $\omega \to \infty$

(B)

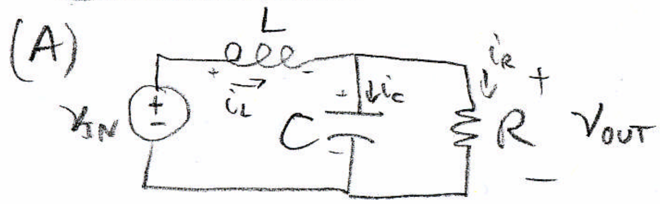
$$\boxed{V_{out} = \frac{V_{in}\left(-\omega^2 L C R_{eq} R_L N \cdot g_m\right.}{\sqrt{\left(1 + \omega^2 R_{eq}^2 (C + C_{gs})^2\right)\left(R_L^2 + \omega^2 N^4 L^2\right)}}}$$

$$\boxed{\phi = 180^\circ - \left[\tan^{-1}\left(\frac{\omega R_{eq}(C + C_{gs})}{1}\right) + \tan^{-1}\left(\frac{\omega L N^2}{R_L}\right)\right]}$$

(D) L and C should be made as large as possible to increase the low frequency gain and extending the bandwidth by decreasing the value of low frequency points for a particular value of gain

Problem 10.4

(A)



KCL at unknown Node:

$$i_L = i_C + i_R$$

with constitutive relations

$$i_C = C\frac{dV}{dt} \quad \text{and} \quad V_L = L\frac{di_L}{dt}$$

where the inductor voltage $V_L = V_{IN} - V_{OUT}$

$$\left(\int \frac{V_L}{L} = C\frac{dV_{OUT}}{dt} + \frac{V_{OUT}}{R}\right)$$

differentiating gives

$$\frac{V_{IN} - V_{OUT}}{L} = C\frac{d^2 V_{OUT}}{dt^2} + \frac{dV_{OUT}}{dt}\cdot\frac{1}{R}$$

OR

$$\boxed{\frac{d^2 V_{OUT}}{dt^2} + \frac{1}{RC}\frac{dV_{OUT}}{dt} + \frac{V_{OUT}}{LC} = \frac{V_{IN}}{LC}}$$

(B) Can be solved using impedances or by substituting $V_{IN} = \operatorname{Re}\{V_{in}e^{j\omega t}\}$ and $V_{OUT} = \operatorname{Re}\{\tilde{V}_o e^{j\omega t}\}$

$$-\omega^2\tilde{V}_o + \frac{j\omega\tilde{V}_o}{RC} + \frac{\tilde{V}_o}{LC} = \frac{V_{in}}{LC}$$

OR

$$\tilde{V}_o\left(-\omega^2 RLC + j\omega L + R\right) = V_{in}\cdot R$$

$$\boxed{\begin{gathered} G \equiv \frac{V_{out}}{V_{in}} = \left|\frac{\tilde{V}_o}{V_{in}}\right| = \frac{R}{\sqrt{(R-\omega^2 RLC)^2 + (\omega L)^2}} \\ \phi = -\tan^{-1}\left(\frac{\omega L}{R(1-\omega^2 LC)}\right) \end{gathered}}$$

Problem 10.4 (continued)

(C) To maximize $G = \frac{R}{\sqrt{R(1-\omega^2 LC) + (\omega L)^2}}$

Need to minimize the denominator by letting $\omega^2 LC = 1$ so $\boxed{C = \frac{1}{\omega^2 L}}$

and $\boxed{G = \frac{R}{\omega L}}$

(D) With $v_{IN}(t) = 0$, the homogeneous response of the damped, second order system will be a sinusoid decaying at $e^{-\alpha t}$ where

$2\alpha = \frac{1}{RC}$ from the differential eqn. in (A)

so

$$\tau = \frac{1}{\alpha} = 2RC \quad \text{with } C = \frac{1}{\omega^2 L} \text{ and } G = \frac{R}{\omega L}$$

$$\tau = \frac{2R}{\omega^2 L} = \frac{2G}{\omega} \qquad \boxed{\tau = \frac{2G}{\omega}}$$

(E) Since the time constant is proportional to the gain, the greater the boost in voltage, the longer the circuit will oscillate after any change in V_{IN}. There is a trade off between response time and voltage gain.