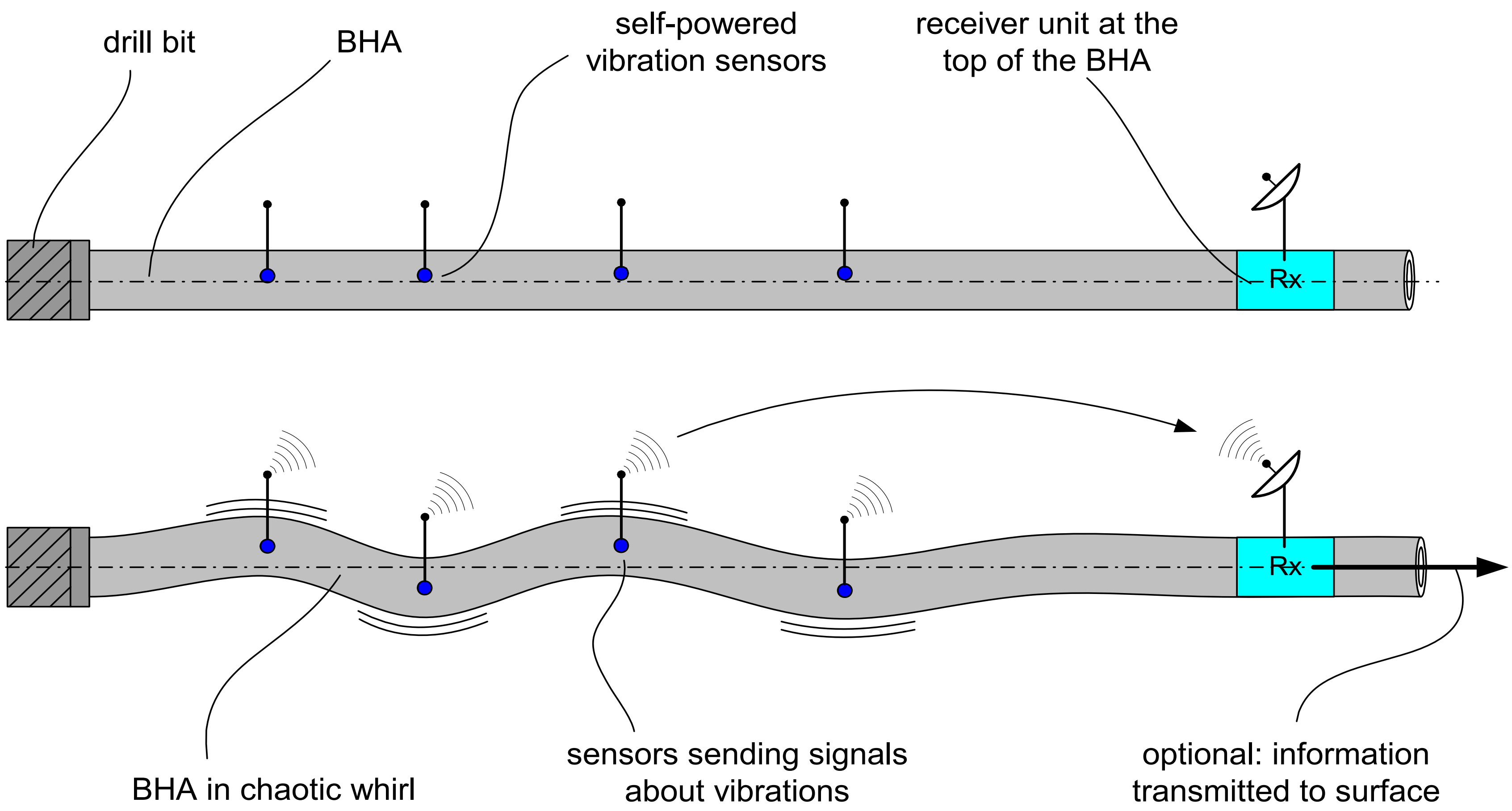
Magnetic-Induction, Vibration Energy Harvesting Device

A Zachary Trimble

Outline and Objectives

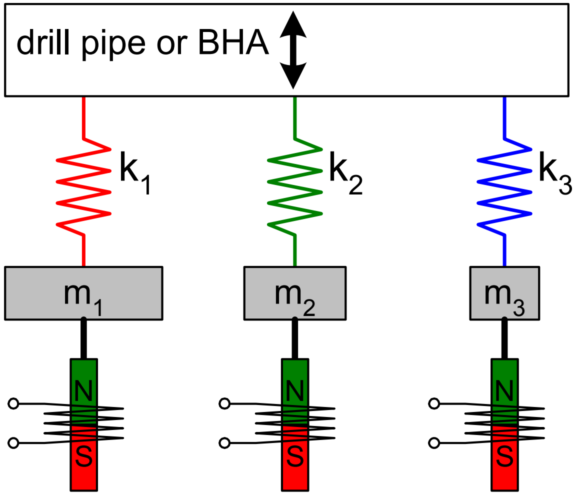
- Project Motivation
  - So we’re all on board
- Planer/annular prototype
  - Informative
- Central prototype
  - Design group input

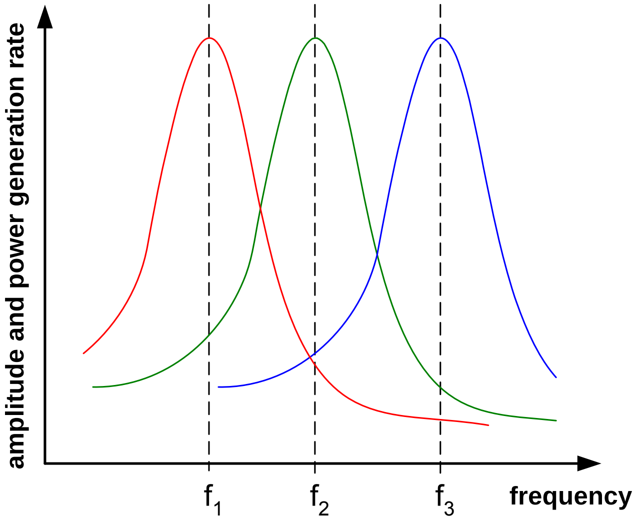
Self-Powered Vibration Monitoring System



Key Idea: Combine Harvesting and Sensing

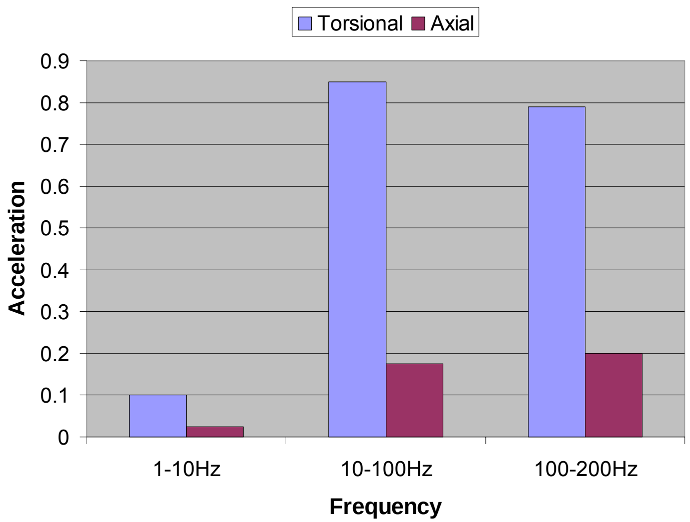
- Drilling Vibrations contain ENERGY and INFORMATION
  - Knowledge of the dynamic characteristics of the vibration harvesting device reveals information about the vibrations itself – REDUCE COMPLEXITY
- Individual tuned mass-spring systems
  - Mechanical frequency spectrum analysis

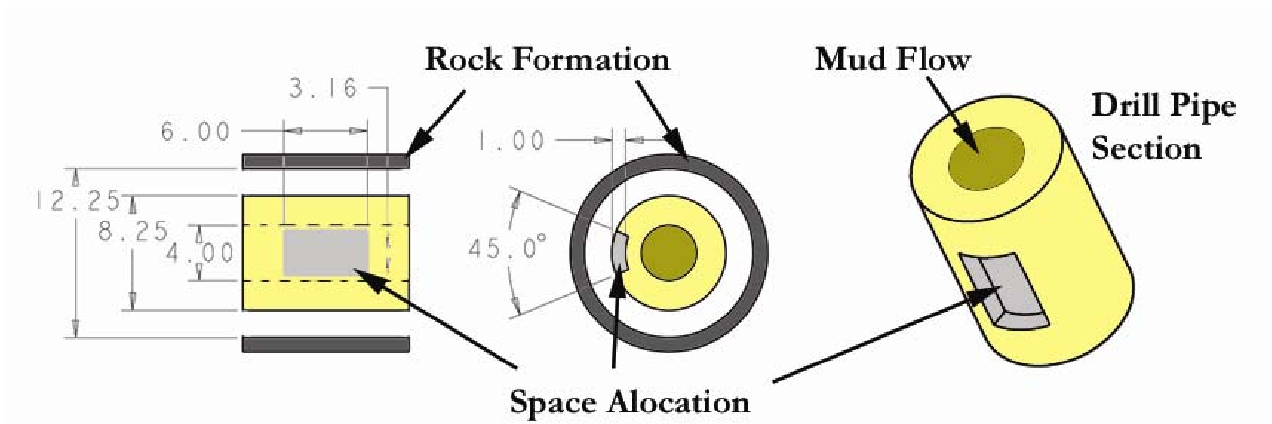




Functional Requirements

Temp.	500	deg F
Press.	30,000	psi
Shock	250	g





Electrical Extraction Methods

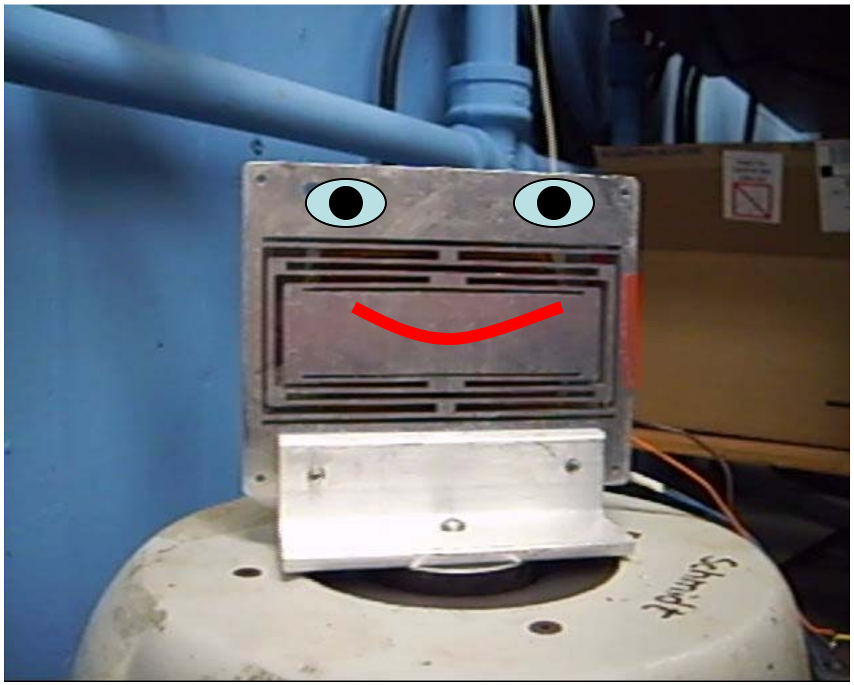
- Electromagnetic Induction
  - Power Needs
  - Range of Motion

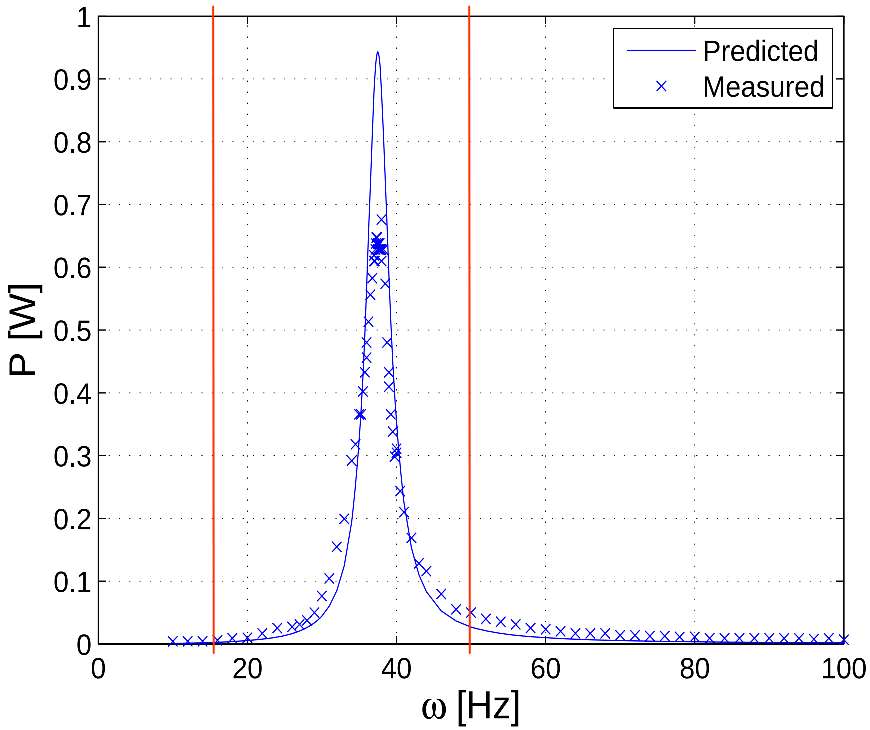
	Variable Capacitance	Piezo Material	Magnetic Induction
Power generation	μW	μW-W	mW-kW
Vibration amplitude	μm	μm	mm-cm
Driving frequency	Any range	Tens of Hz	Any range
Ease of system design	Difficult	Easy	Easy
Cost	High	High	Modest
Lifetime	Low	High	High

Table 2.3: Comparison of energy harvesting strategies.

Jonnalagadda, Aparna S, (2007), "Magnetic Induction Systems to Harvest Energy from Mechanical Vibrations", MIT SM Thesis, January 2007.

Overview





$$P = \frac{mQ_i A_n^2}{4\omega_n}$$

Prototype Device;

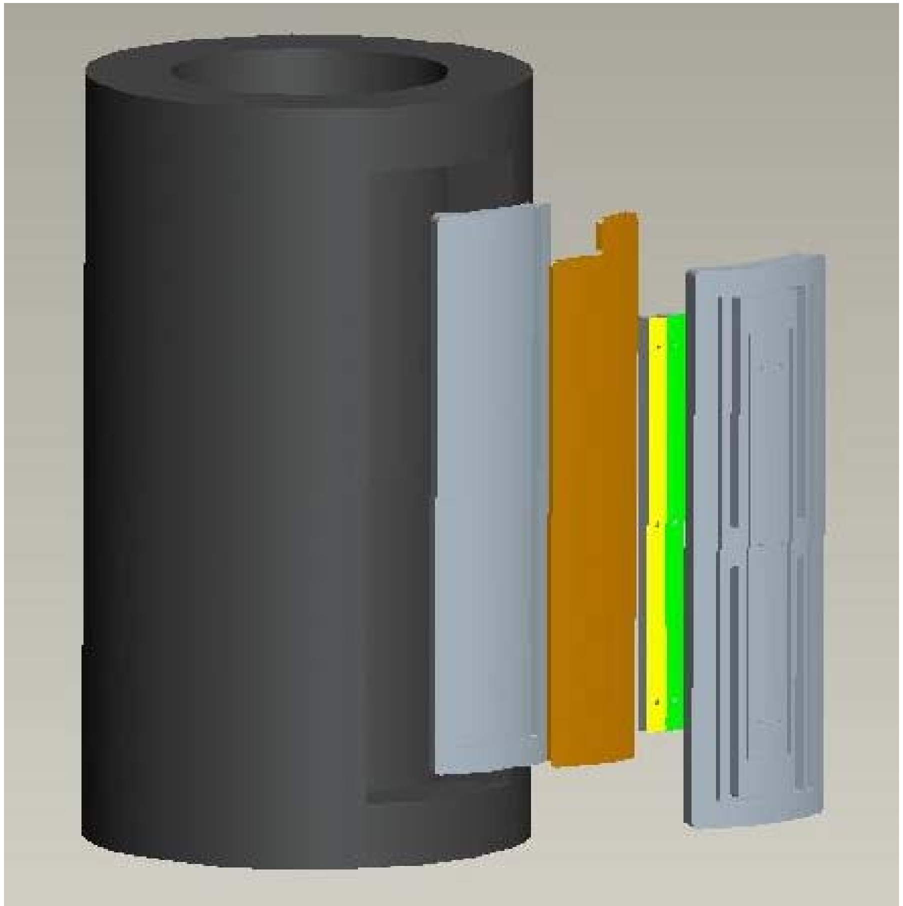
$P_{estimate} = 0.67$ W

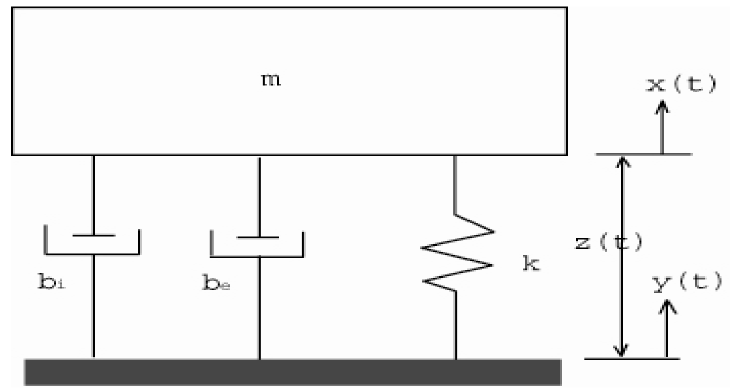
$P_{actual} = 0.6$W

Estimated Device;

P = 1W

First Order Model





First Order Model

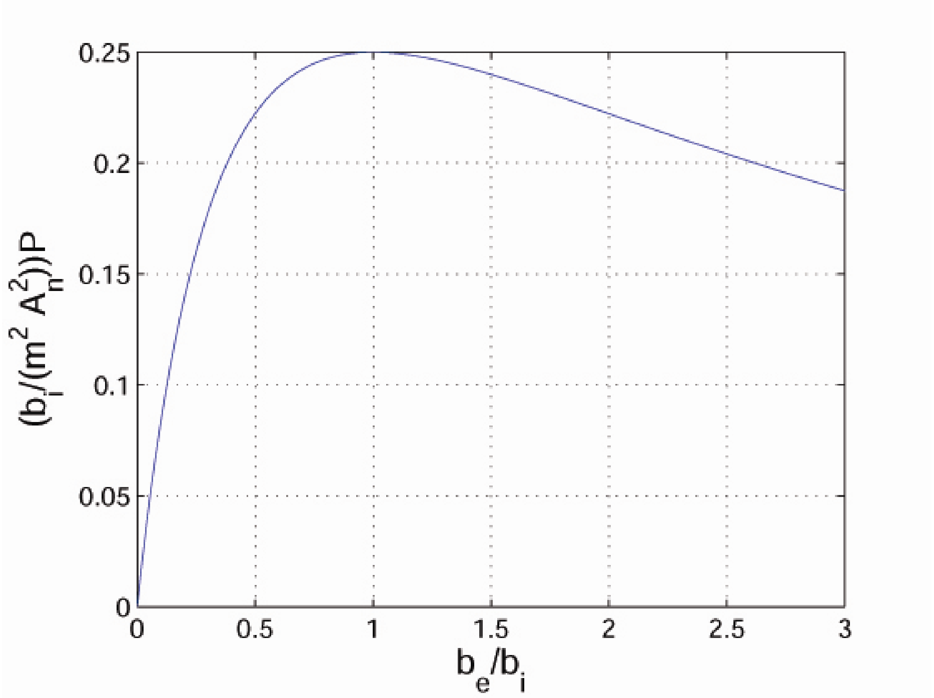
• Governing Equation:	$m\ddot{z}(t)+(b_i+b_e)\dot{z}(t)+kz(t)=-m\ddot{y}(t)$
• Normalized Governing Equation:	$\ddot{z}(t)+2\omega_n(\varsigma_i+\varsigma_e)\dot{z}(t)+\omega_n^2 z(t)=-\ddot{y}(t)$
• Assume harmonic input:	$y(t)=Ye^{j(\omega t)}$
• Resonant Solution:	$\dot{z}(t)=\dfrac{\ddot{Y}}{2\omega_n(\varsigma_i+\varsigma_e)}e^{j(\omega_n t-\pi)}$
• Power	$P=Fv=bv^2=b_e\ddot{z}^2$ $P=\dfrac{m^2\ddot{Y}^2 b_e}{2(b_i+b_e)^2}=\dfrac{m\ddot{Y}^2}{32\omega_n\varsigma_e}$

Matched Damping

$$P = \frac{m^2 \ddot{Y}^2 b_e}{2(b_i + b_e)^2}$$

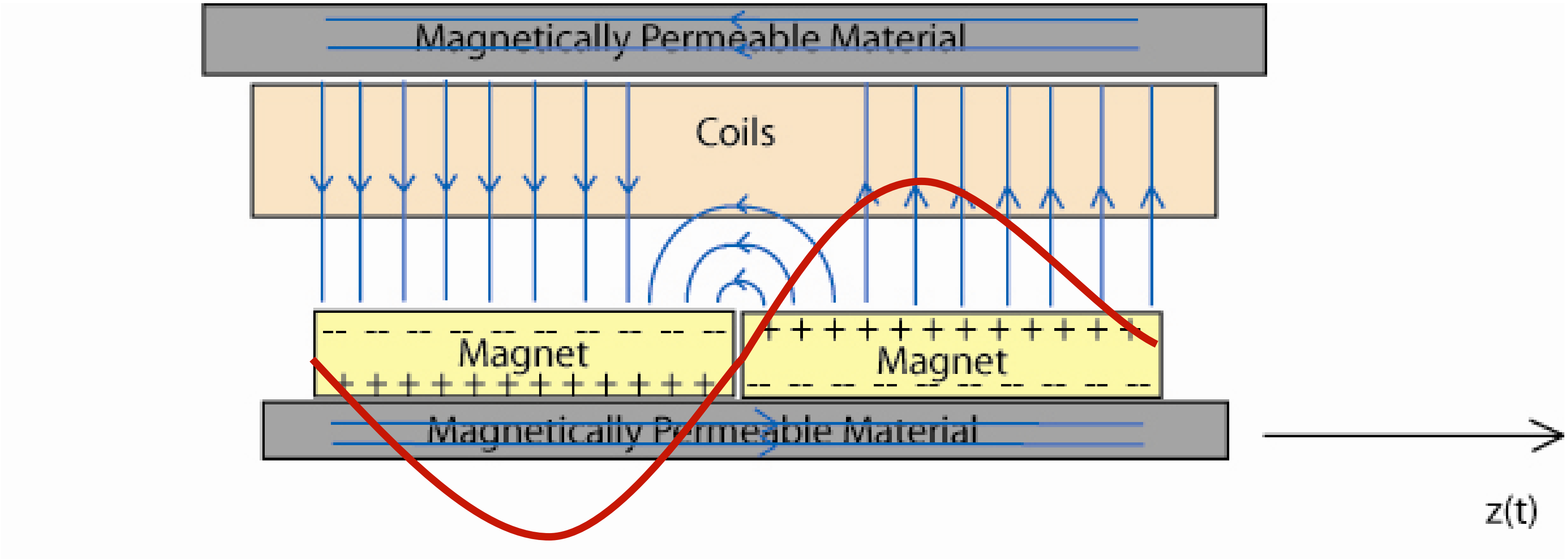
$$\frac{dP}{db_e} = \frac{m\ddot{Y}^2}{2} \frac{b_i - b_e}{(b_i + b_e)^3} \Rightarrow b_i = b_e$$

$$P = \frac{m^2 \ddot{Y}^2}{8b_i} = \frac{m\ddot{Y}^2}{16\omega_n \varsigma_i} = \frac{m\ddot{Y}^2 Q_i}{8\omega_n}$$



Physical amplitude limits bi

Electromagnetic Induction Geometry



$$\lambda = \lambda_0 \sin\left(\pi z_n\right)$$

Coil Design: Damping Factor

$$P_{max} = b\dot{z}^2(t) = \frac{u^2\lambda_0^2\pi^2 N}{8RW_m^2}\dot{z}^2(t) \text{ for } N > 2$$

u = total number of turns

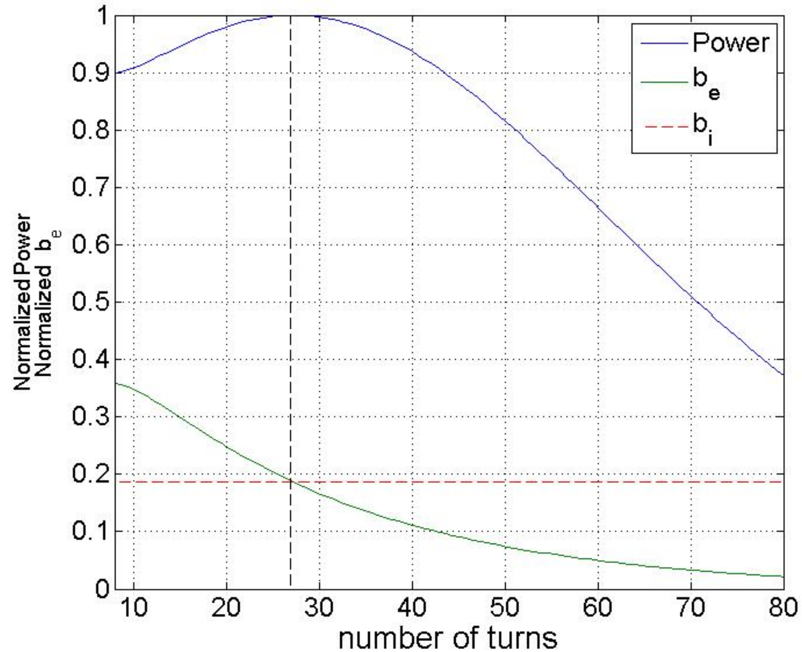
λ = magnetic flux

N = number of phases

R = Coil resistance

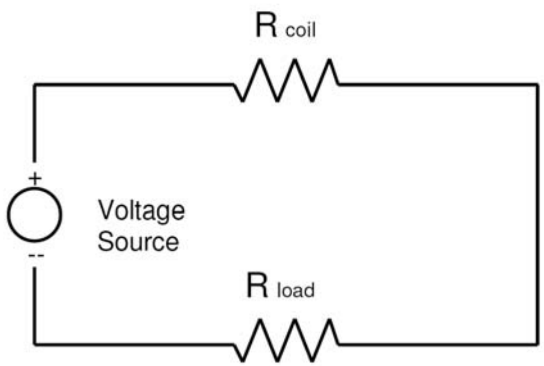
W_m = Magnet Width

NOTE: Only free parameter is number of turns



Electromagnetic Voltage/Power

$$V = \frac{d\lambda}{dt} = \frac{d\lambda}{dz}\frac{dz}{dt} = \frac{d\lambda}{dz}\dot{z}(t)$$



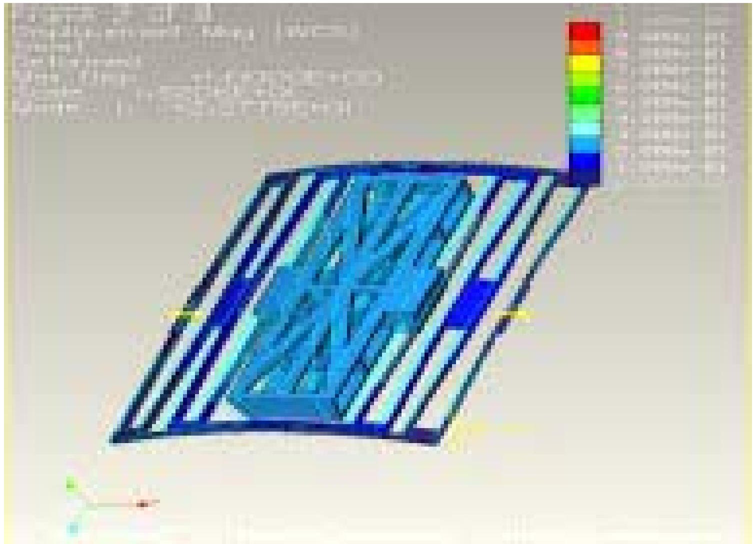
$$V_L = \frac{R_L}{R_c + R_L} V_n$$

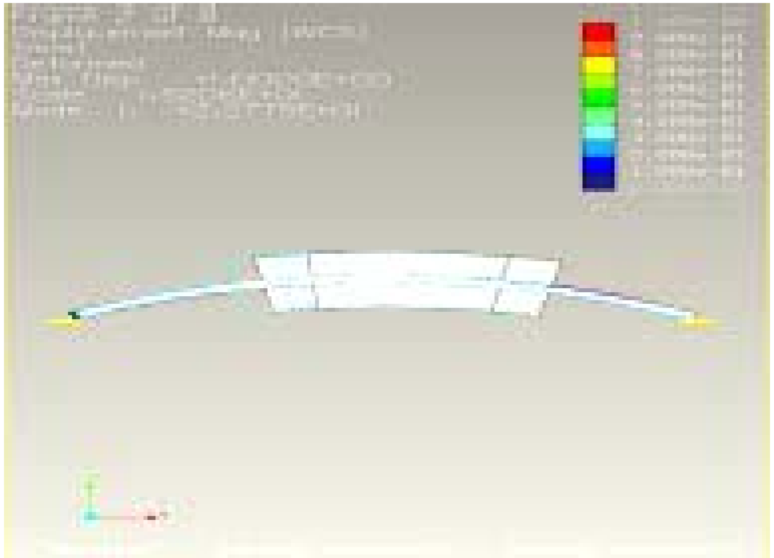
$$P = V_L \cdot i_L = \frac{V_L^2}{R_L}$$

$$P_{max} = \frac{V_n^2}{4R_c}$$

$$P_{max\ total} = \sum_{n=1}^{N} \frac{V_n^2}{4R_n}$$

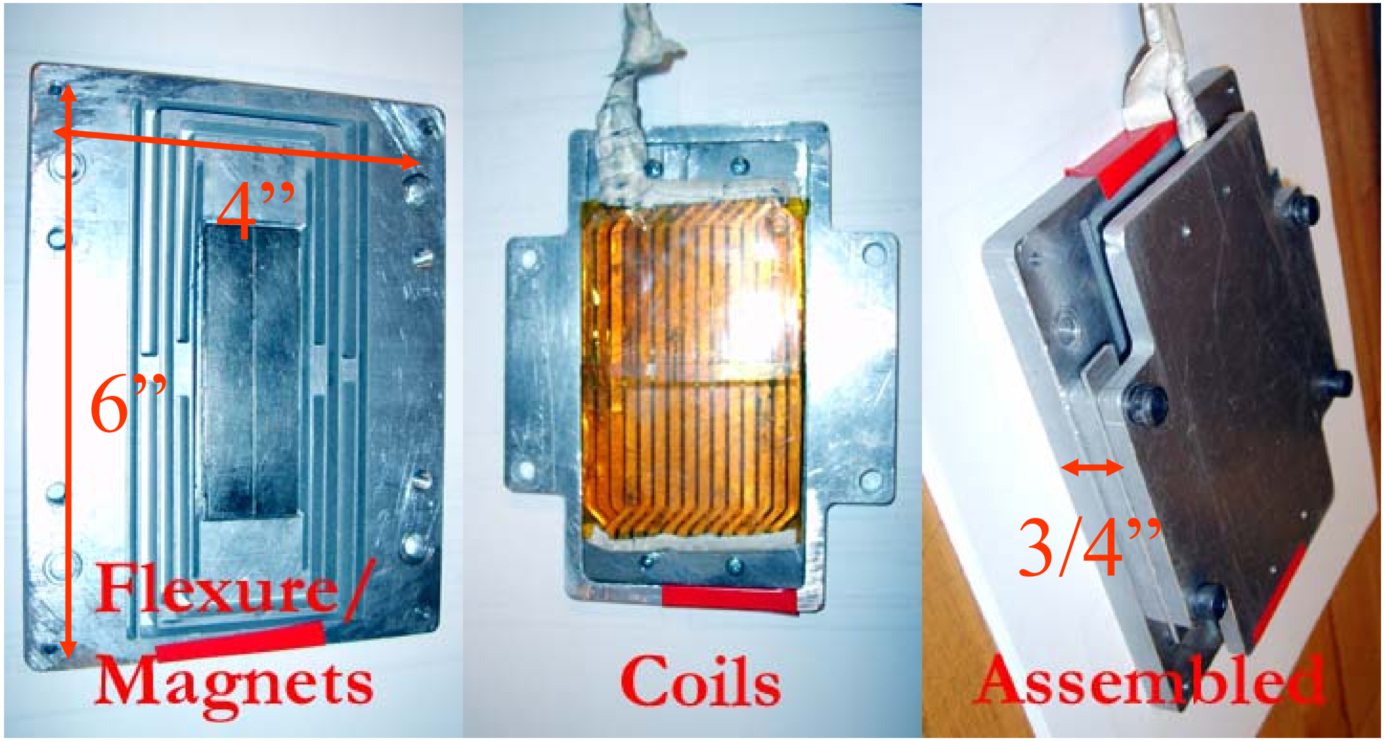
FEA verification



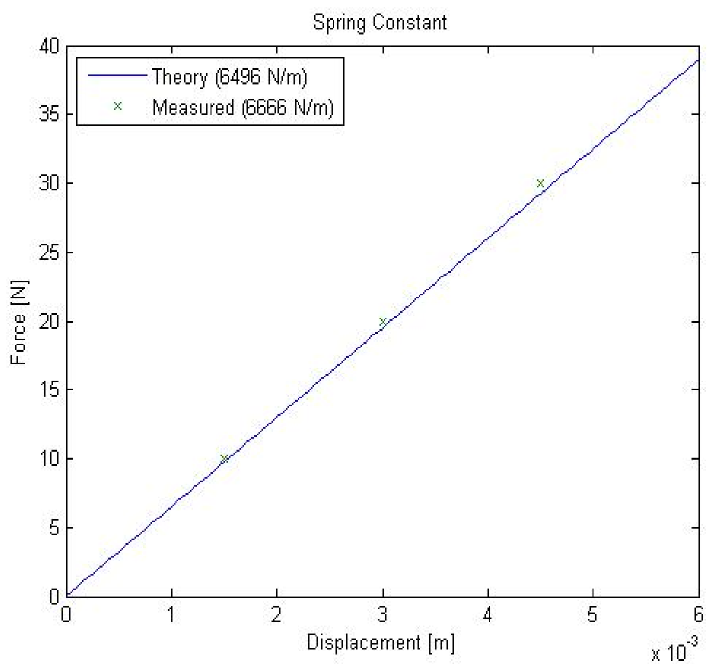


Out of plane motion minimal—sets air gap

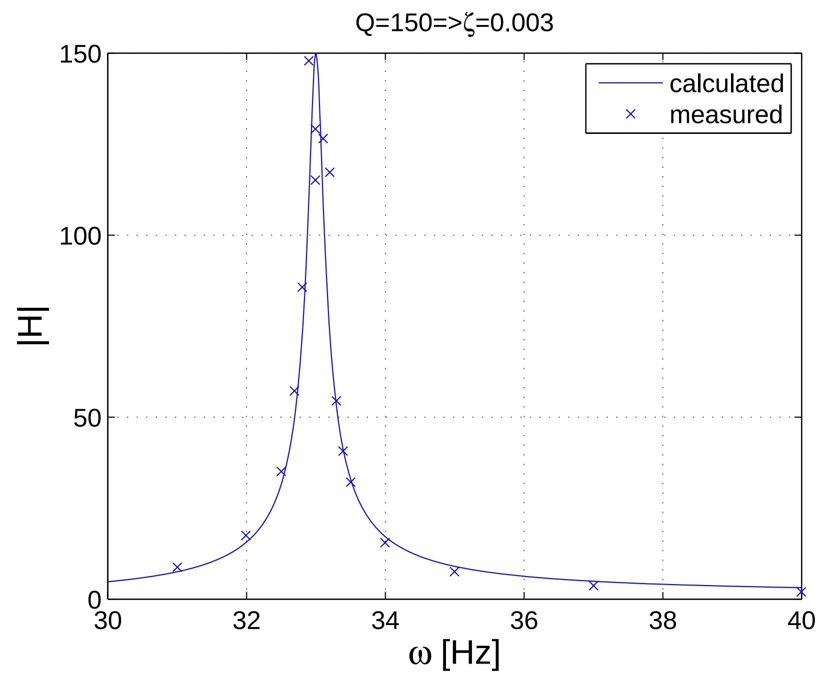
Existing Prototype



Spring Constant

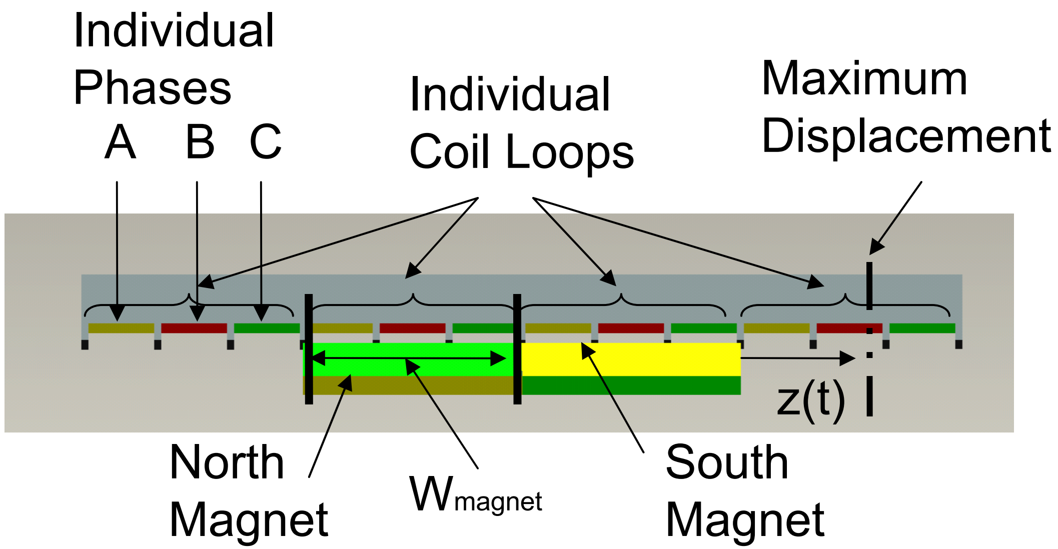


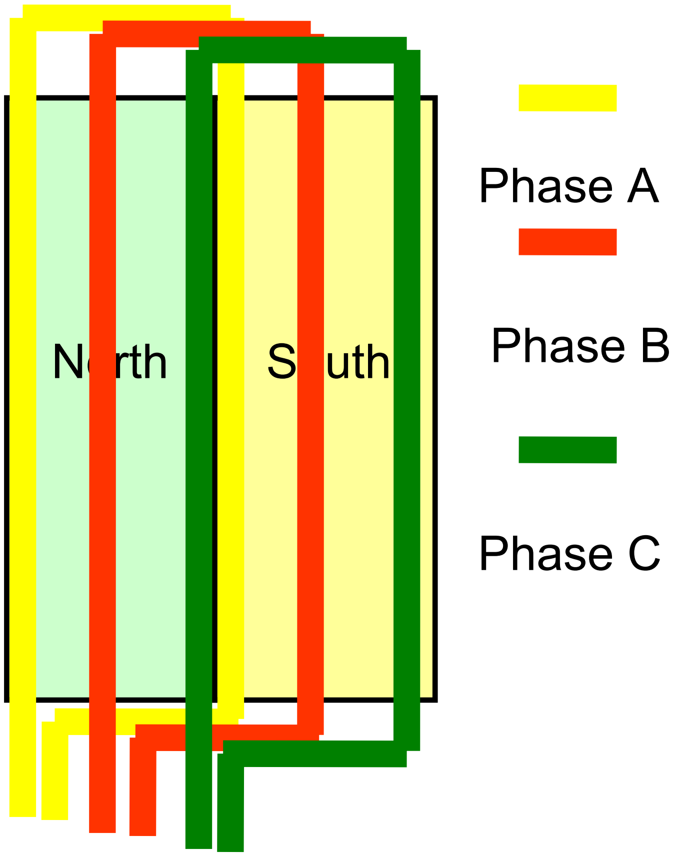
2% error



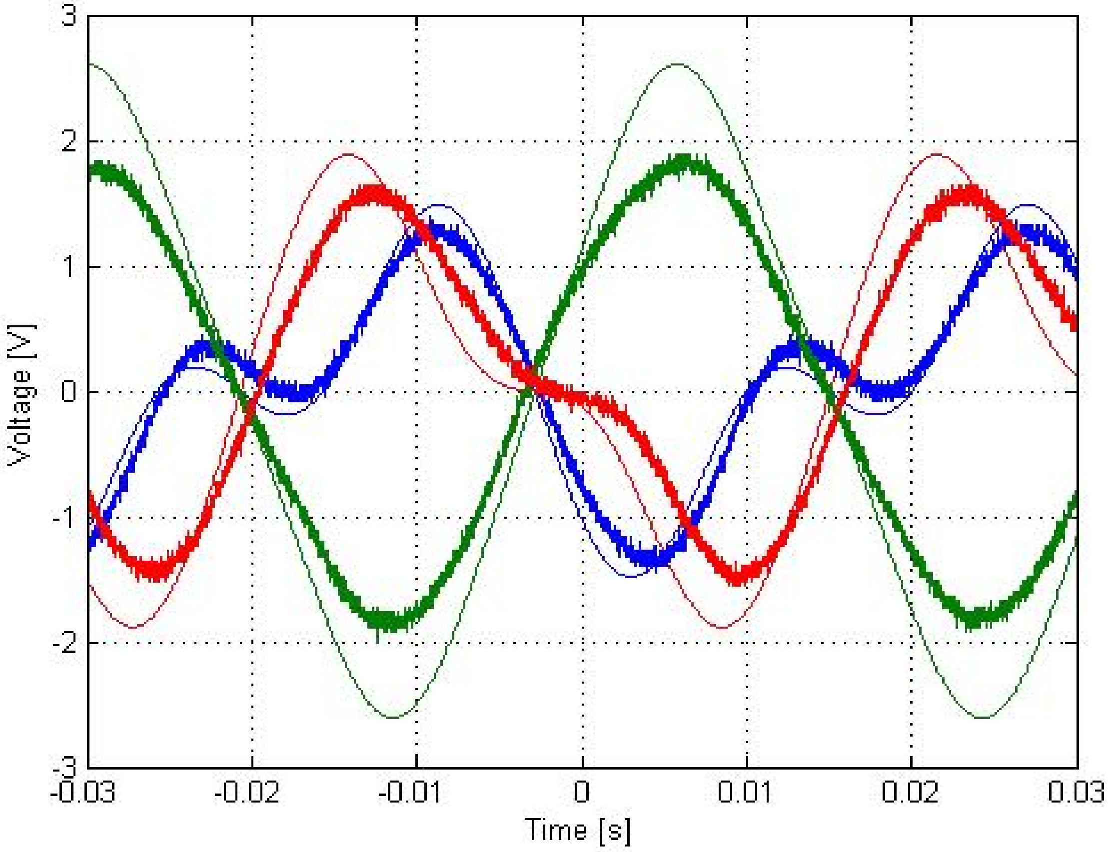
Q = 150

Electromagnetic Coil Geometry

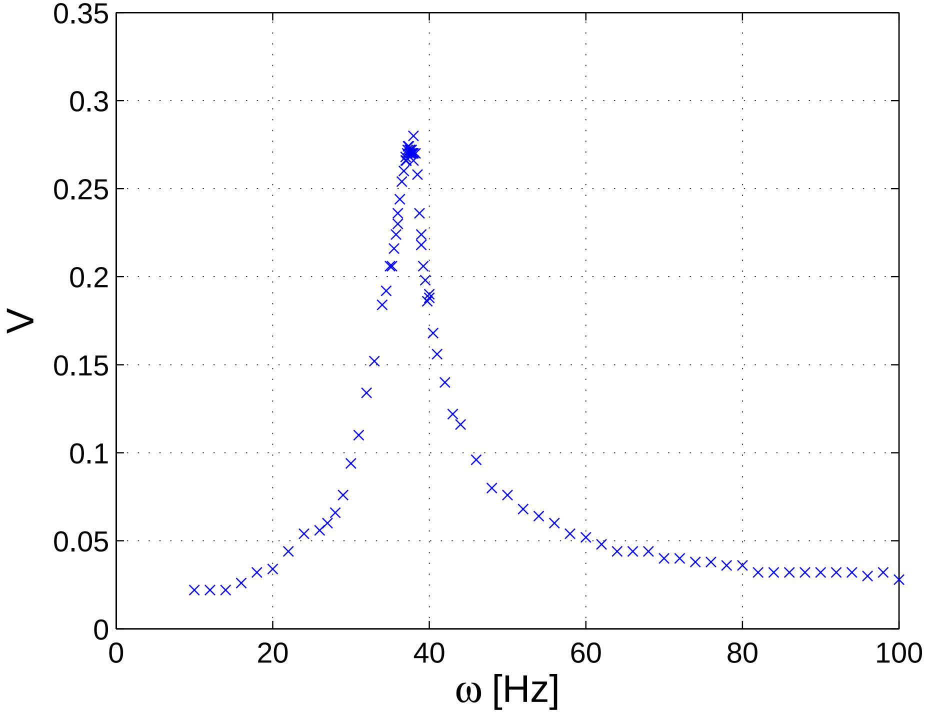


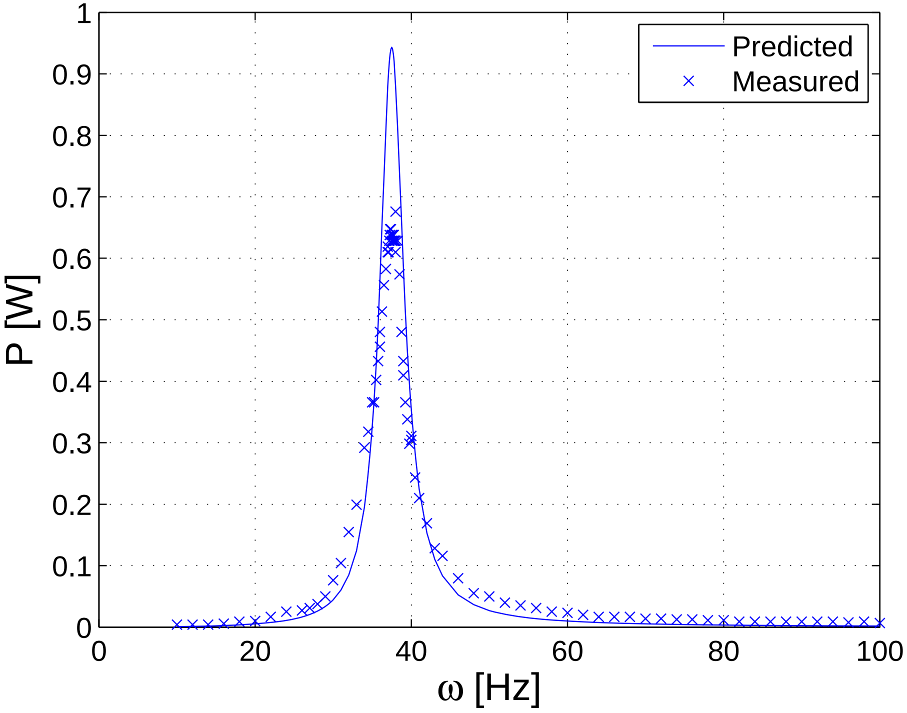


Voltage vs Time

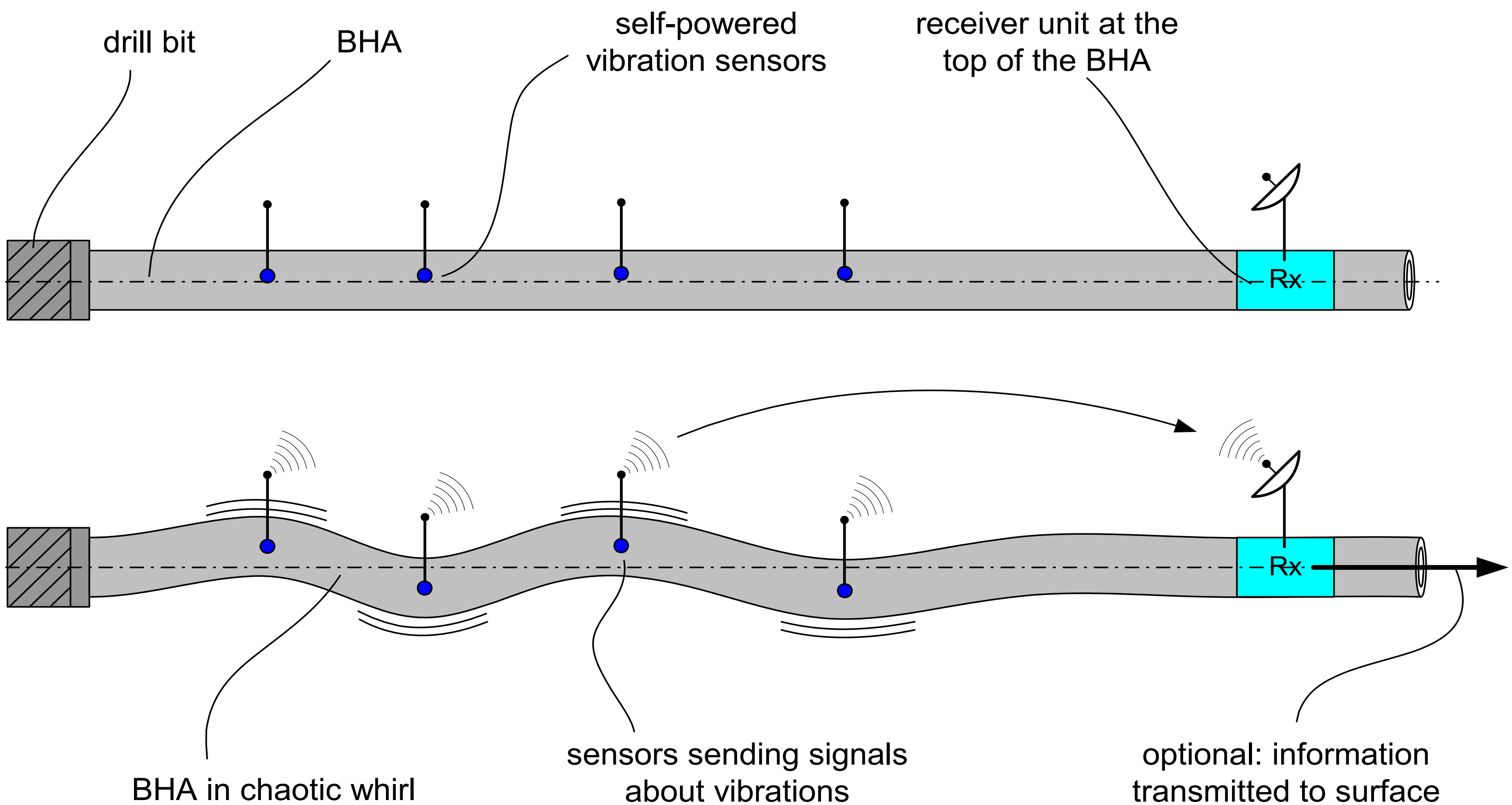


Voltage/Power





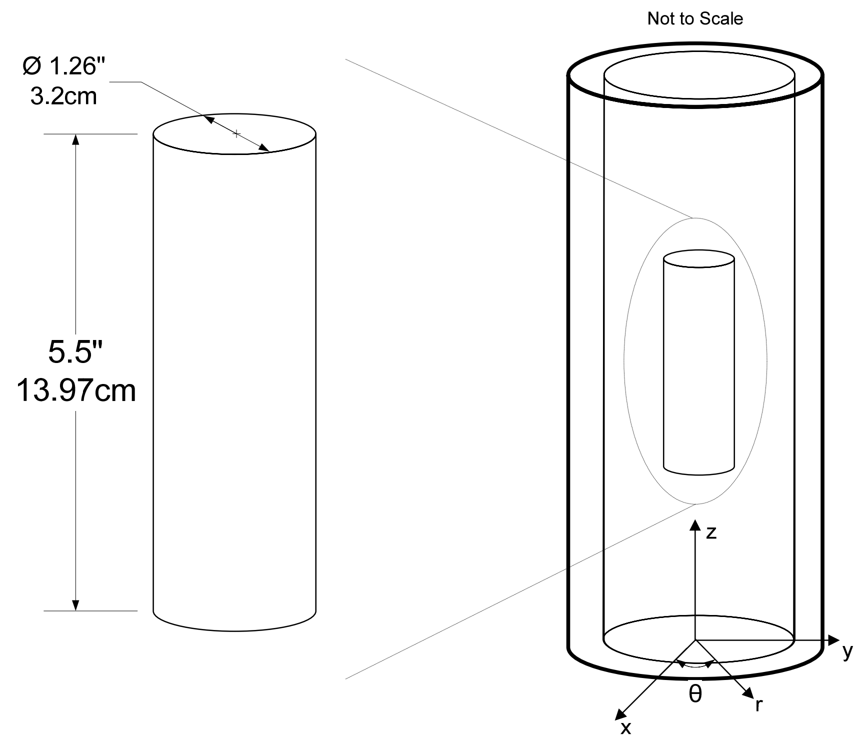
Self-Powered Vibration Monitoring System



Review

•Design a Vibration Energy Harvesting Device that will fit in the space and size allocation shown, and provide as much power as possible when subjected to accelerations similar to those provided by the Stonehouse facility.

Space and Size Allocation



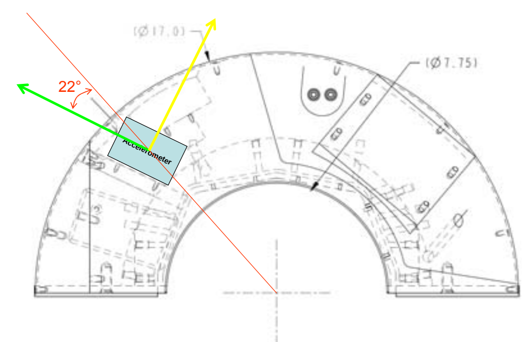
Maximum Harvester size including all circuitry and casings

Expected harvester location in the center of the pipe/tool.

Small Intermission for REALITY

First Order Power Estimate Vibration Input

- Data is not aligned with typical (r,θ,z) coordinates
- To improve estimate rotate coordinate system 22°
  - Rotation angle is determined by protractor measurement on the shown scale drawing

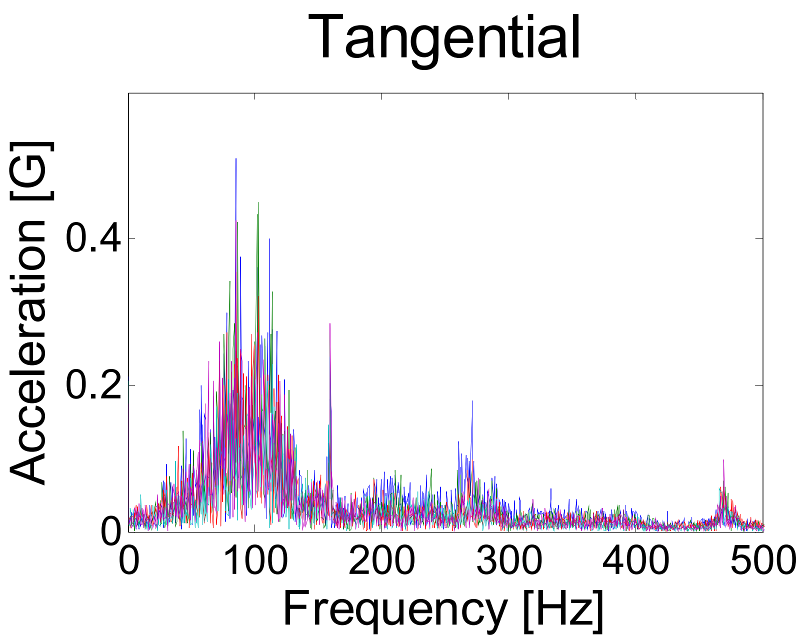


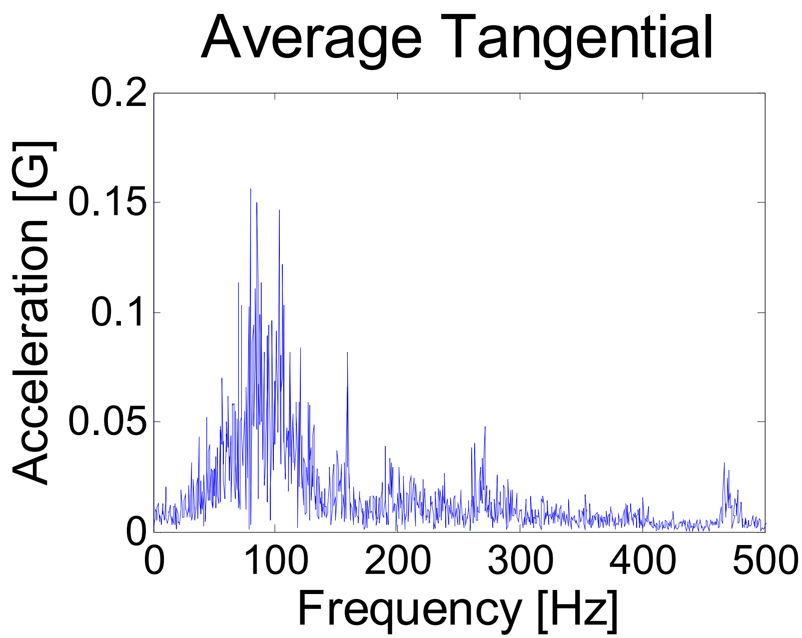
Cross Section of Surface Sub

$$A_\theta = A_y \cos(22^\circ) - A_g \sin(22^\circ)$$

$$A_r = A_y \sin(22^\circ) + A_g \cos(22^\circ)$$

First Order Power Estimate Vibration Input

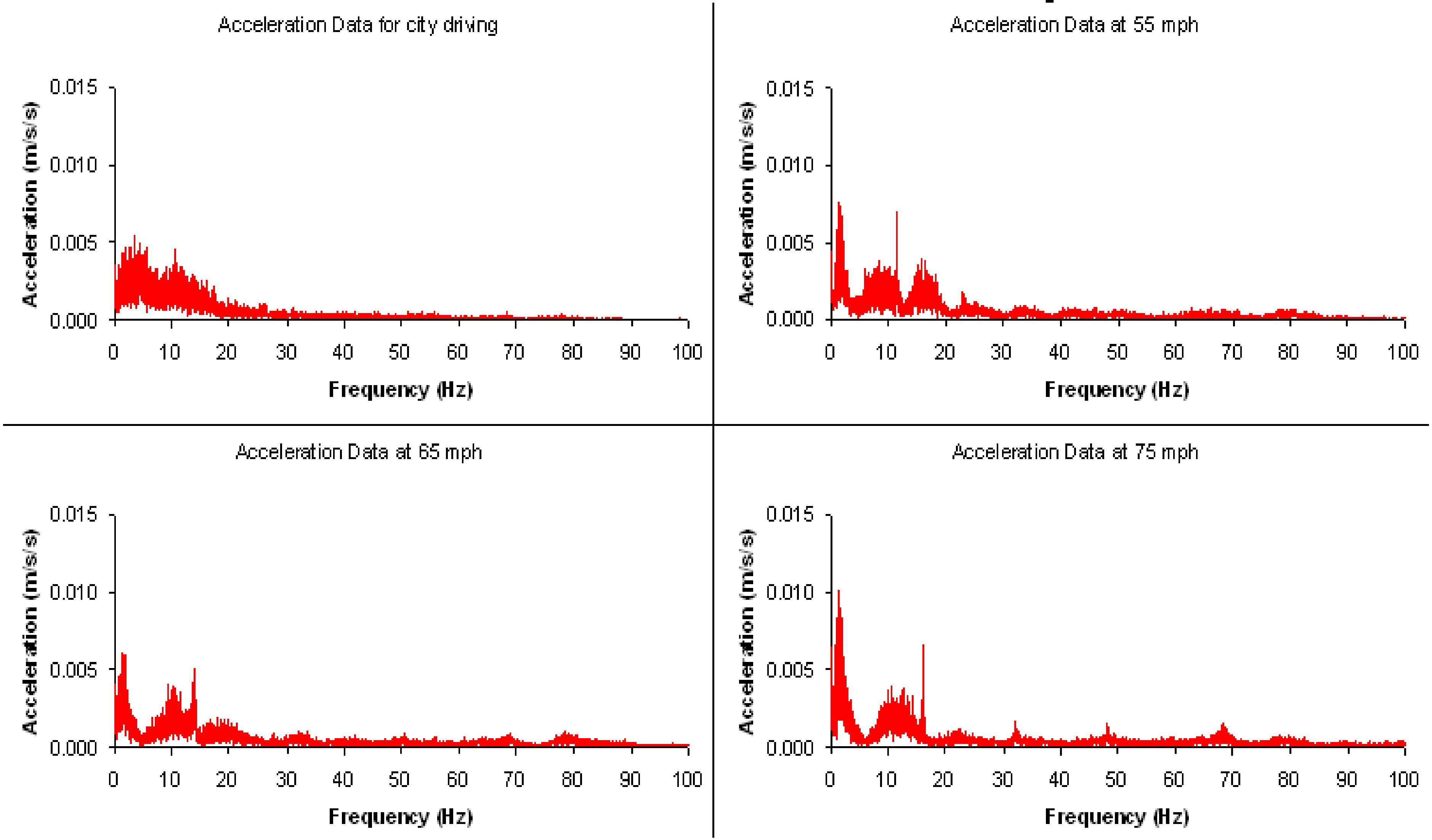




Data collection

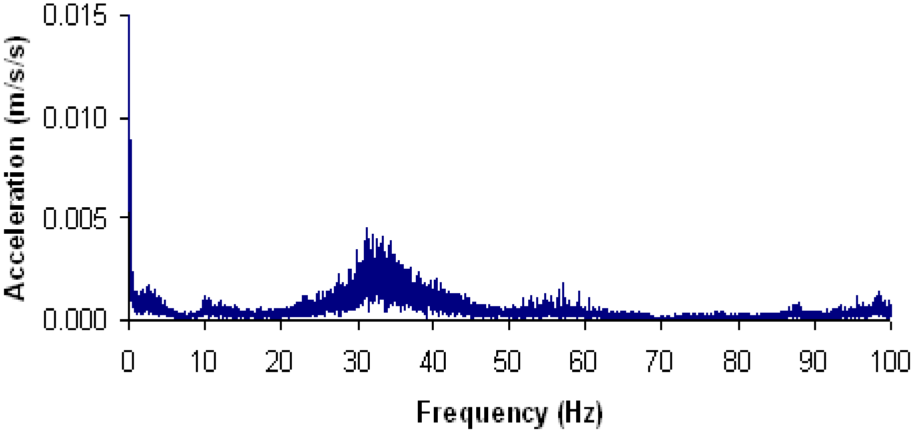
- 3-axis accelerometer mounted on Nissan Altima car door
- or

Acceleration Data – Up and

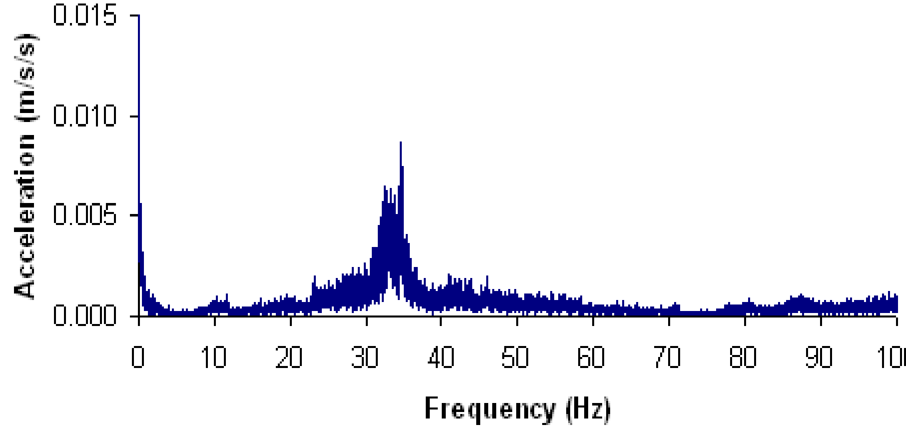


Acceleration Data – Side to side

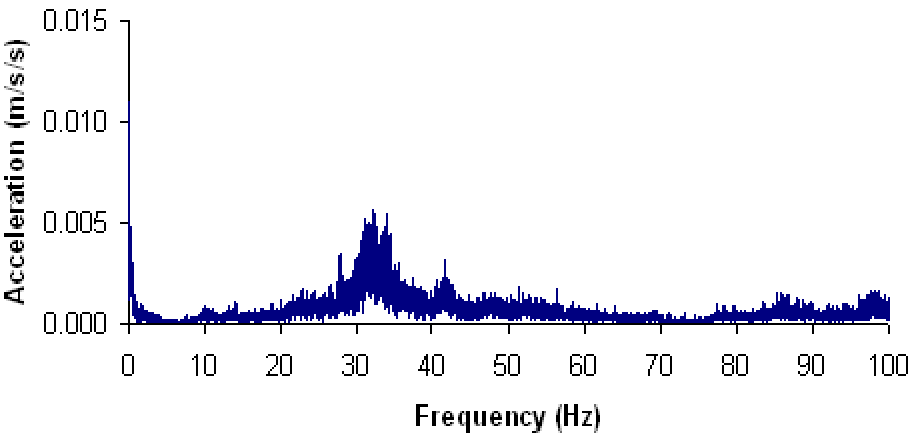
Acceleration Data for city driving



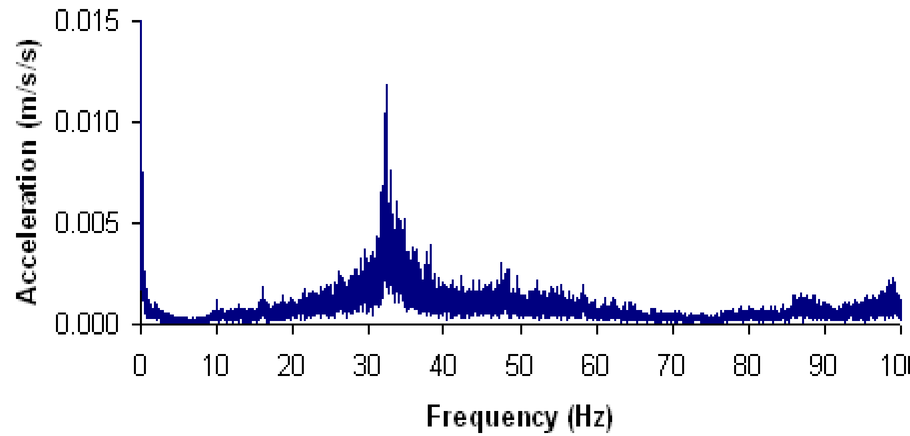
Acceleration Data at 55 mph



Acceleration Data at 65 mph

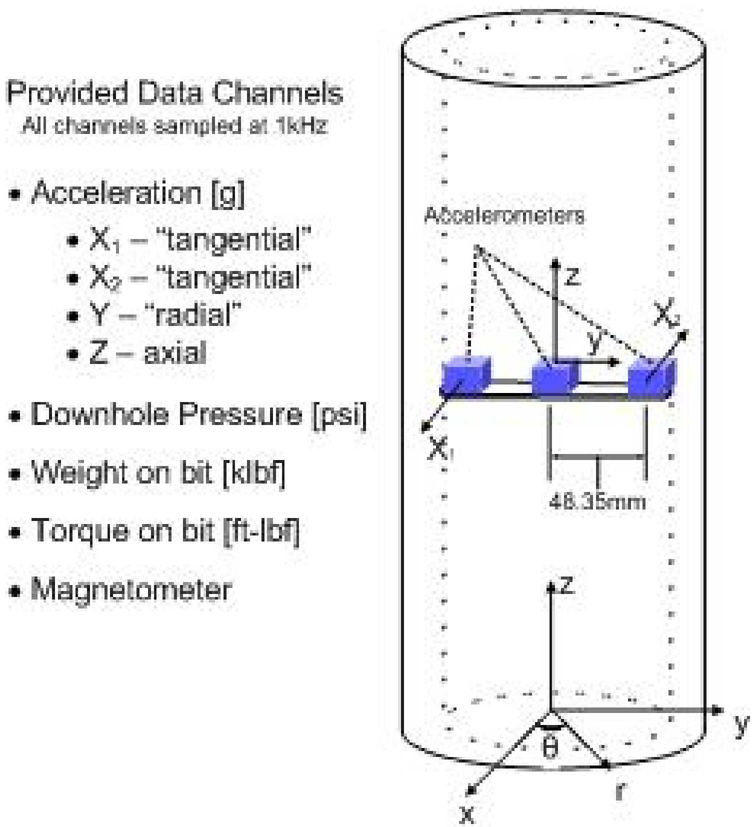


Acceleration Data at 75 mph



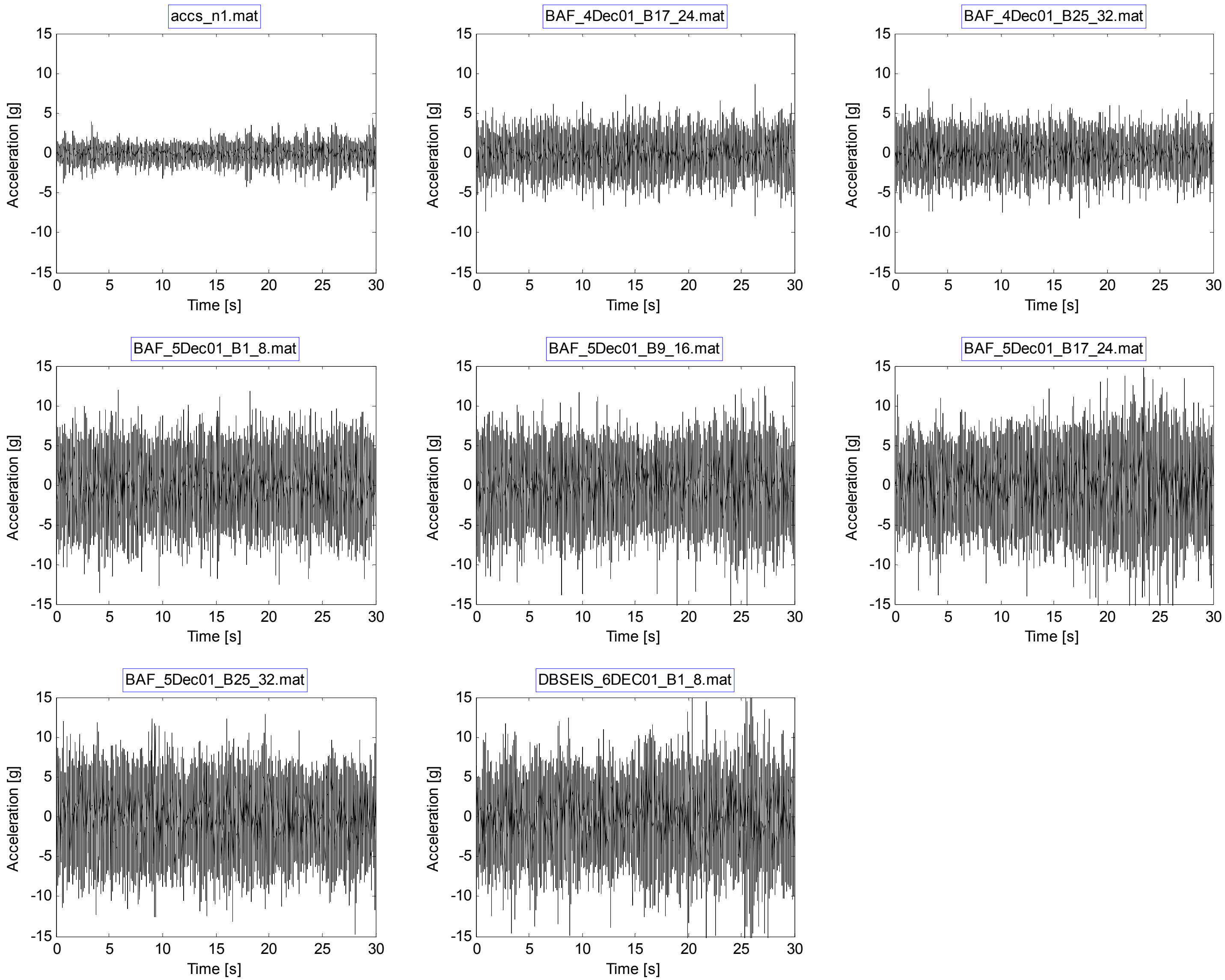
Data

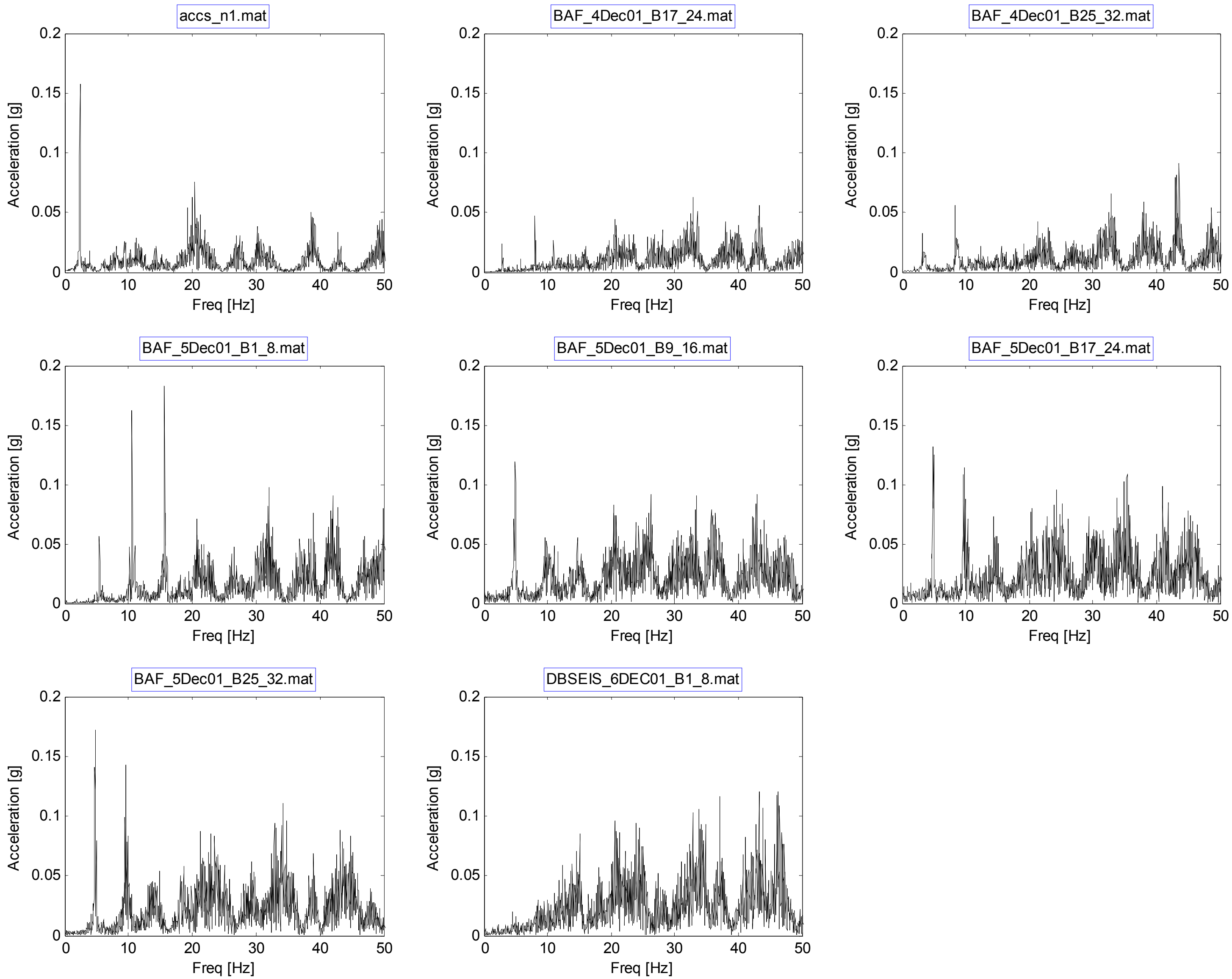
Not to Scale



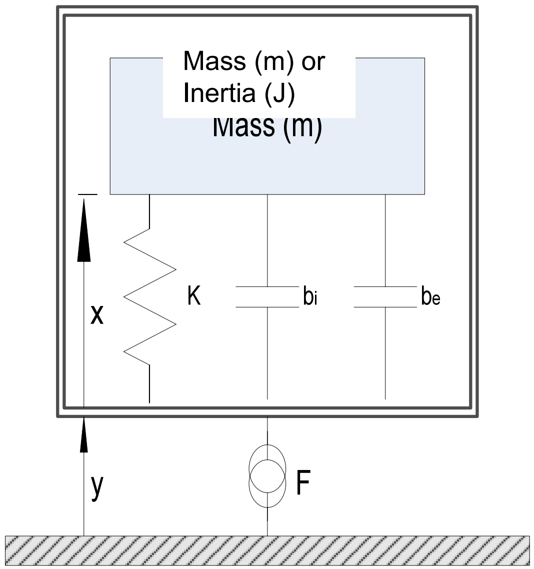
- Data is taken on two different tools (Labeled BAF and DBSEIS)
  - For BAF tool data is taken in 36s intervals
  - For DBSEIS tool data is taken in 27s intervals
- The data is a combination of resting, rotating but no downpressure, and drilling (rotating and downpressure).
- For future simulations, the "active" sections of data were extracted from the complete records
- z-acceleration is measured directly
- Tangential acceleration (α) is calculated as (where r=48.35mm)

$$\alpha = \frac{x_1 + x_2}{2r}$$


accs_n1.mat
BAF_4Dec01_B17_24.mat
BAF_4Dec01_B25_32.mat
BAF_5Dec01_B1_8.mat
BAF_5Dec01_B9_16.mat
BAF_5Dec01_B17_24.mat
BAF_5Dec01_B25_32.mat
DBSEIS_6DEC01_B1_8.mat
Acceleration [g]
Time [s]
15
10
5
0
-5
-10
-15
0
5
10
15
20
25
30


accs_n1.mat
BAF_4Dec01_B17_24.mat
BAF_4Dec01_B25_32.mat
BAF_5Dec01_B1_8.mat
BAF_5Dec01_B9_16.mat
BAF_5Dec01_B17_24.mat
BAF_5Dec01_B25_32.mat
DBSEIS_6DEC01_B1_8.mat
Acceleration [g]
Freq [Hz]
0
0.05
0.1
0.15
0.2
0
10
20
30
40
50

Models



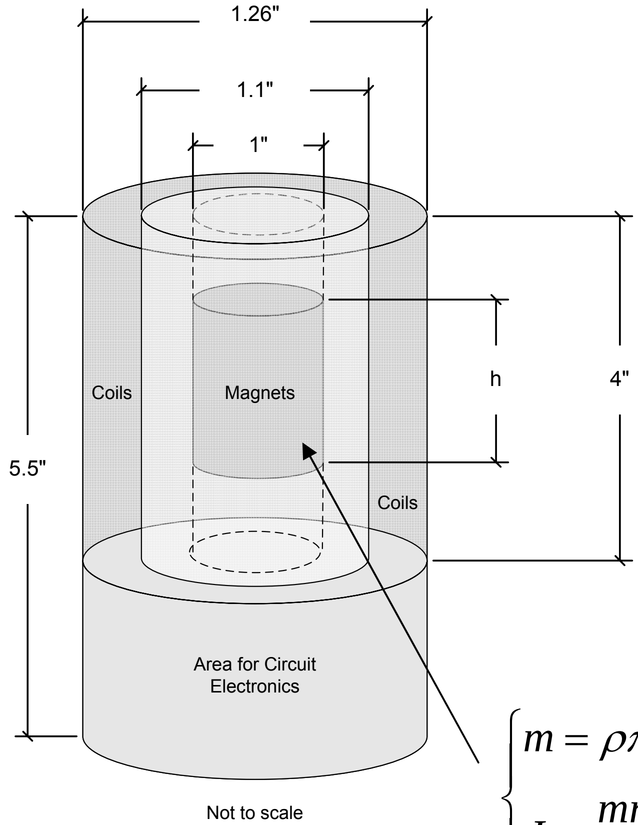
$$\ddot{x} + \frac{(b_i + b_e)_x}{m}\dot{x} + \frac{K_x}{m}x = -\ddot{y}$$

$$P = (b_e)_x \dot{x}^2$$

$$\ddot{\phi} + \frac{(b_i + b_e)_\phi}{J}\dot{\phi} + \frac{K_\phi}{J}\phi = -\ddot{\theta}$$

$$P = (b_e)_\phi \dot{\phi}^2$$

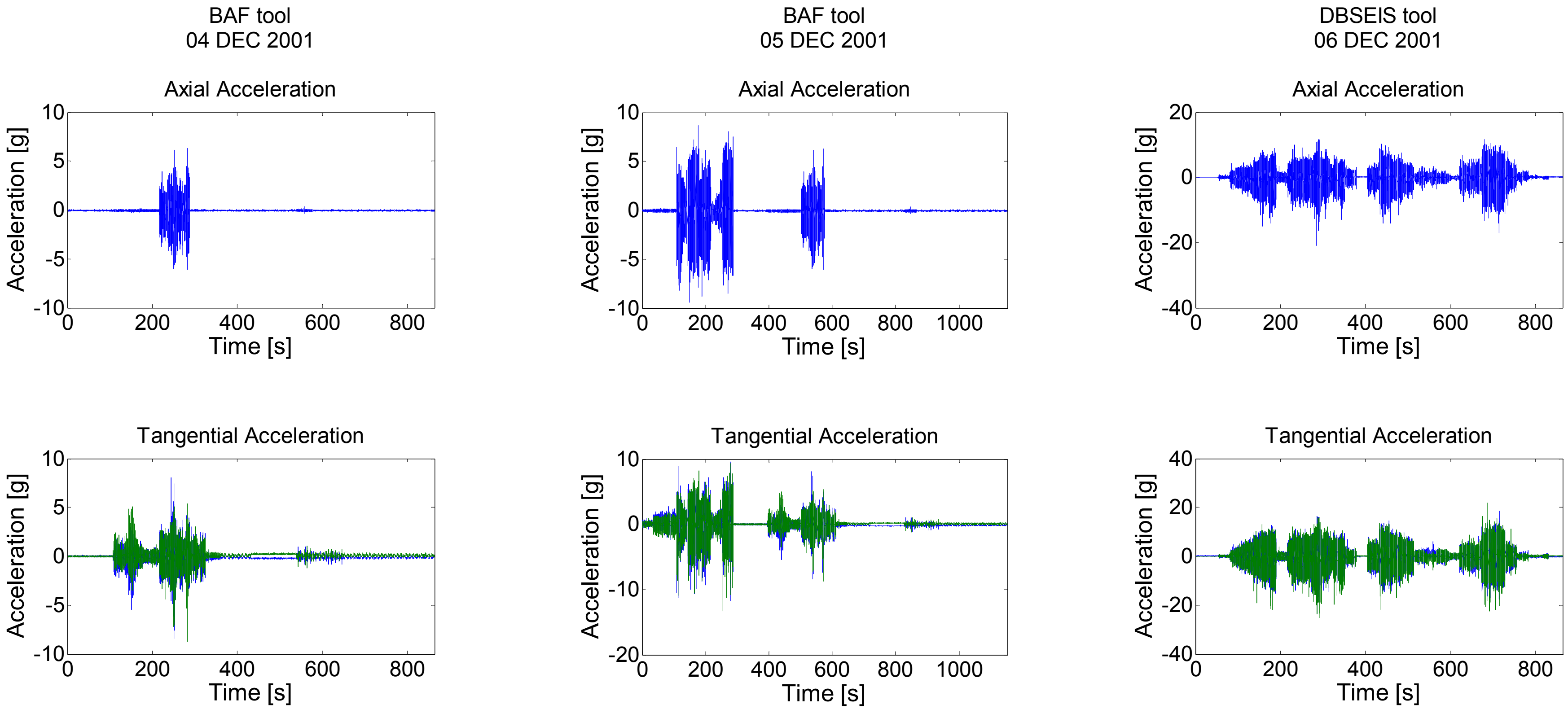
Parameters



$$\begin{cases} m = \rho \pi r^2 h \\ J = \dfrac{mr^2}{2} \end{cases}$$

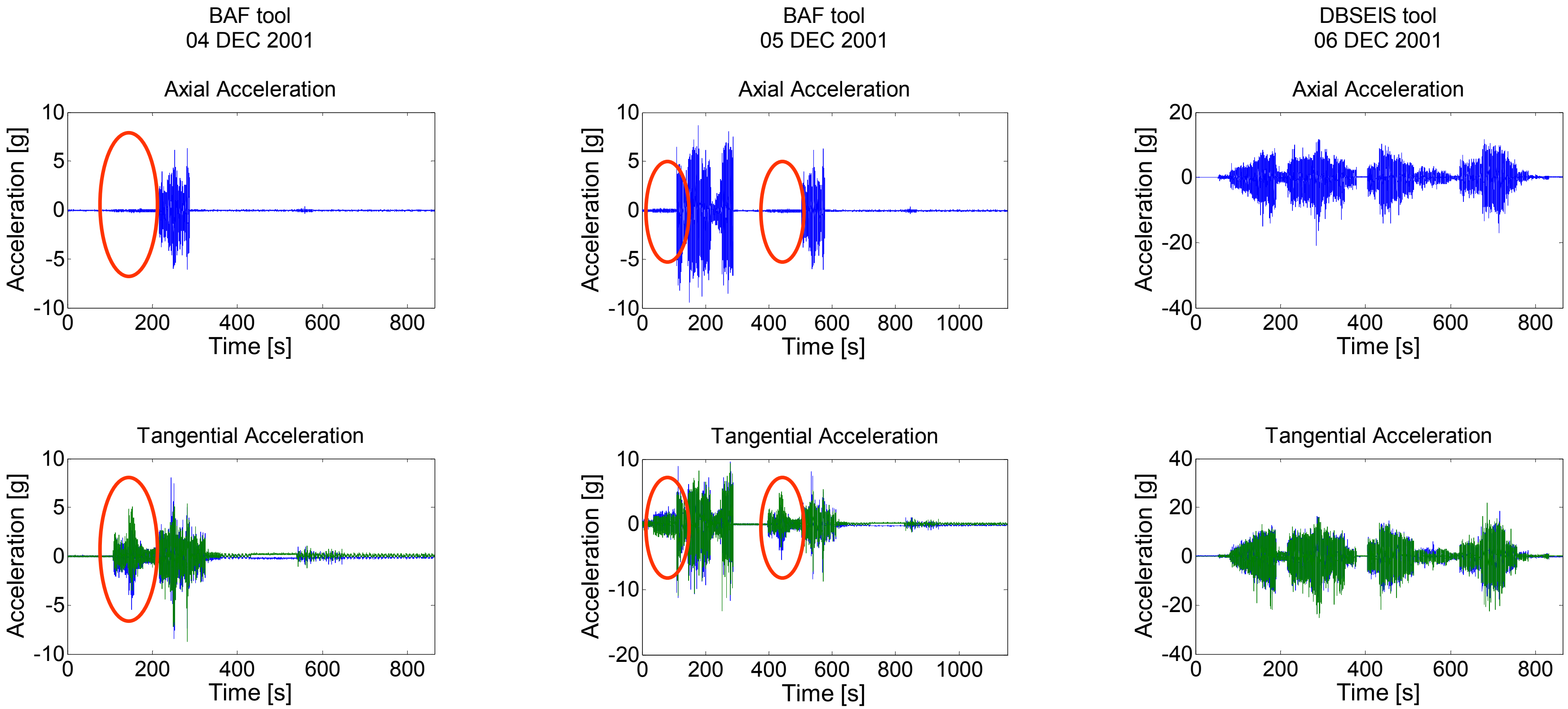
- Density/Inertia of moving mass is combination of steel core and magnets
  - ρ(steel)~7.8g/cc
  - ρ(magnet)~7.4g/cc
    - www.kjmagnetics.com

Raw Data

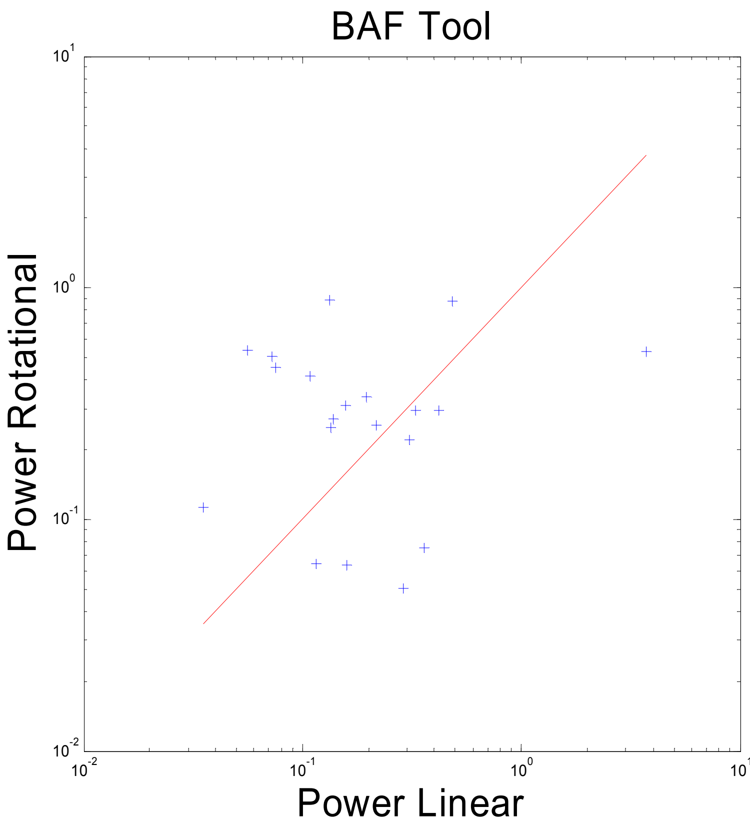


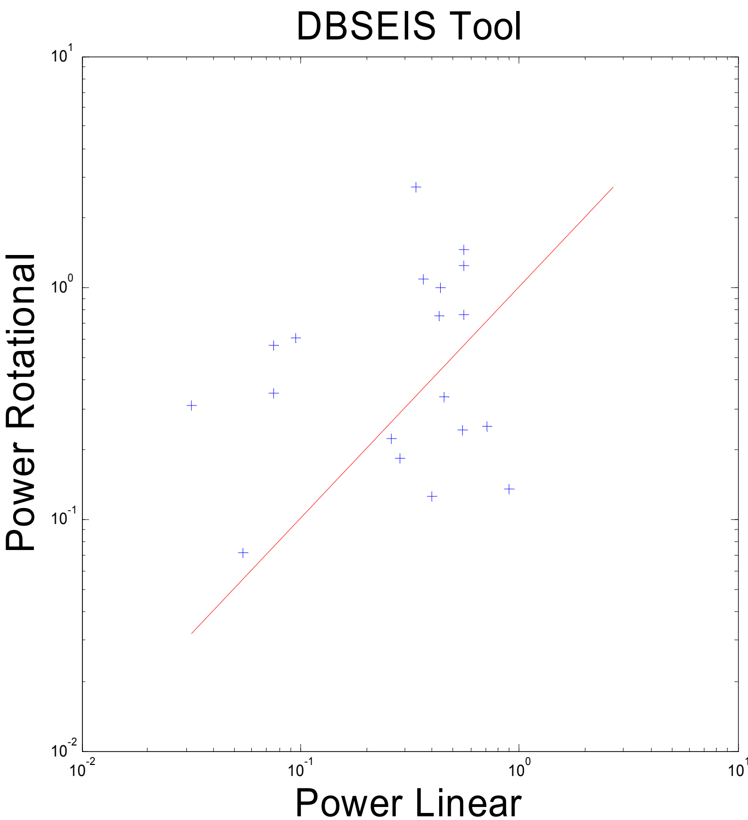
BAF tool
04 DEC 2001
BAF tool
05 DEC 2001
DBSEIS tool
06 DEC 2001
Axial Acceleration
Tangential Acceleration
Acceleration [g]
Time [s]

Rotational vs Linear



Comparison over all traces





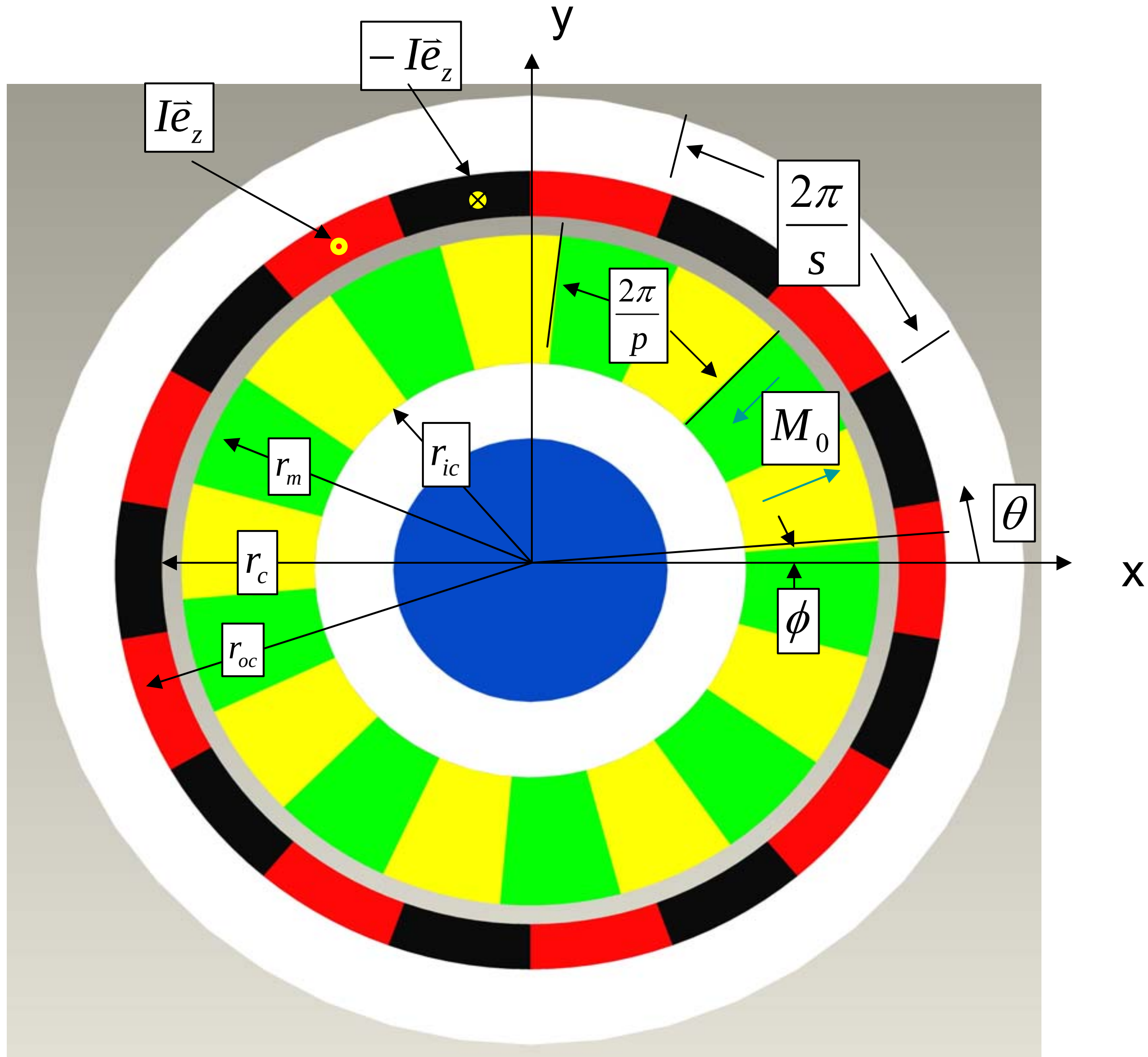
Ideal Model

s = number of current pole pairs

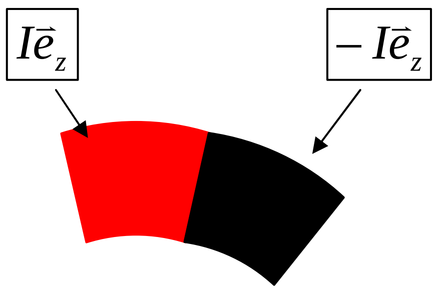
p = number of magnet pole pairs

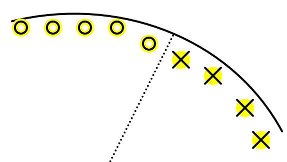
θ = rotational coordinate (measured from the x-axis

Φ = rotor position

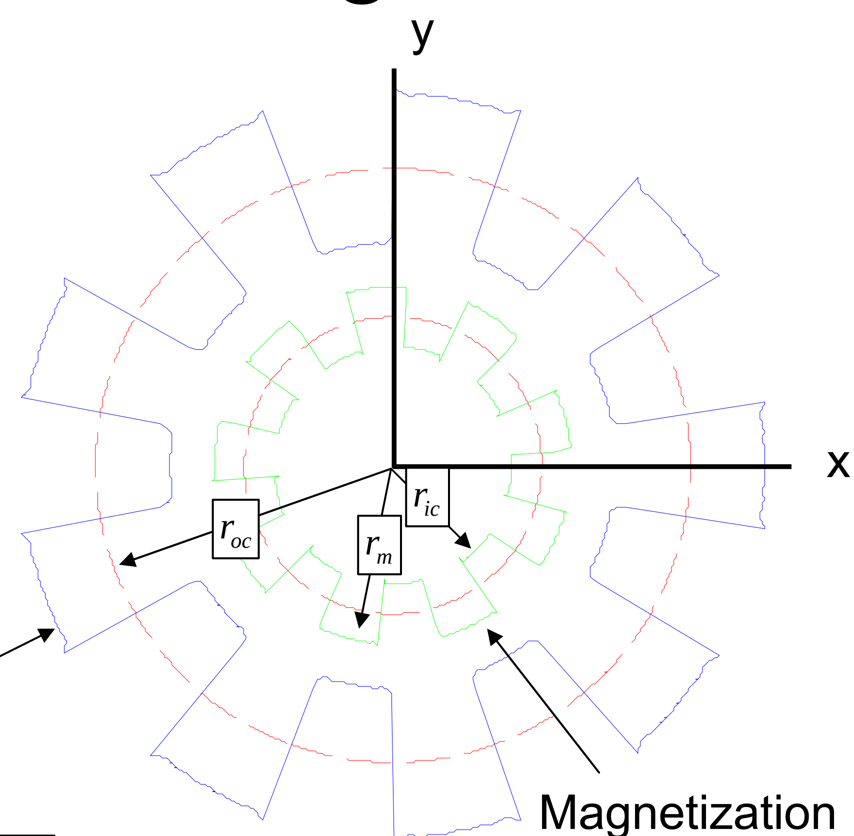


Simplification – Surface Current and Magnetic Charge





$$K_0' = \frac{current}{length} = \frac{1}{3}\frac{I}{r_{oc}\pi/s} = \frac{Is}{3\pi r_{oc}}$$



$$K_i = -\frac{4}{\pi}K_0'\sum_k \frac{(-1)^{(k)}\cos\left[s(2k-1)\left(\theta - \frac{2\pi(i-2)}{3}\right)\right]}{2k-1}$$

$$M = \frac{4}{\pi}M_0\sum_k \frac{\sin[s(2k-1)(\theta-\phi)]}{2k-1}$$

i represents the phase

Governing Equations

Ampere's Law $\nabla X\vec{H} = \vec{J} = 0$

Gauss' Law $\nabla \cdot \vec{B} = \nabla \mu_0 \vec{H} = 0$

$$\left.\begin{aligned} \nabla X\vec{H} &= \vec{J} = 0 \\ \nabla \cdot \vec{B} &= \nabla \mu_0 \vec{H} = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} \vec{H} &= -\nabla \varphi \\ \nabla^2 \varphi &= 0 \end{aligned}$$

Laplace's equation

Boundary Conditions:

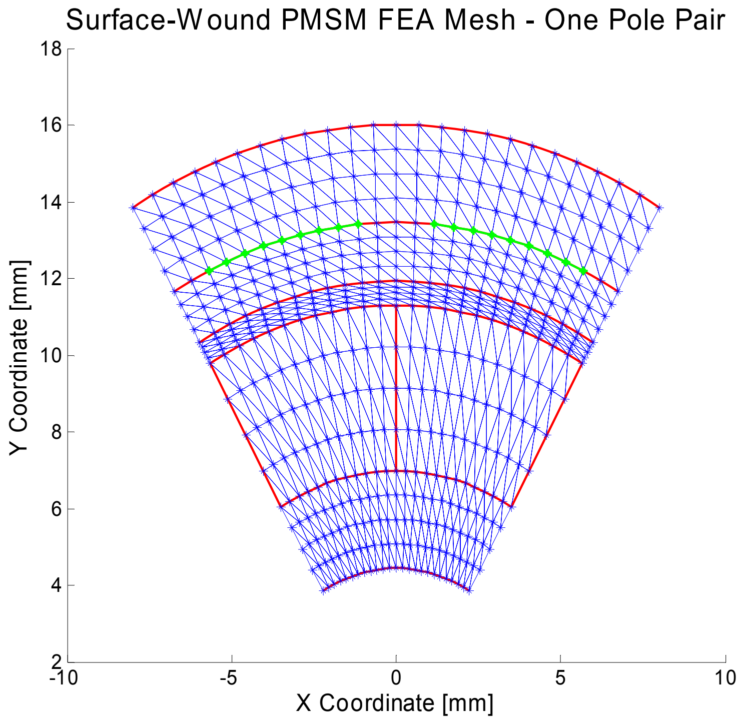
$$r = r_{oc} \quad H_\theta\big|_{r=r_{oc}} = \frac{1}{r_{oc}} \frac{\partial \varphi}{\partial \theta}\bigg|_{r=r_{oc}} = -K_z$$

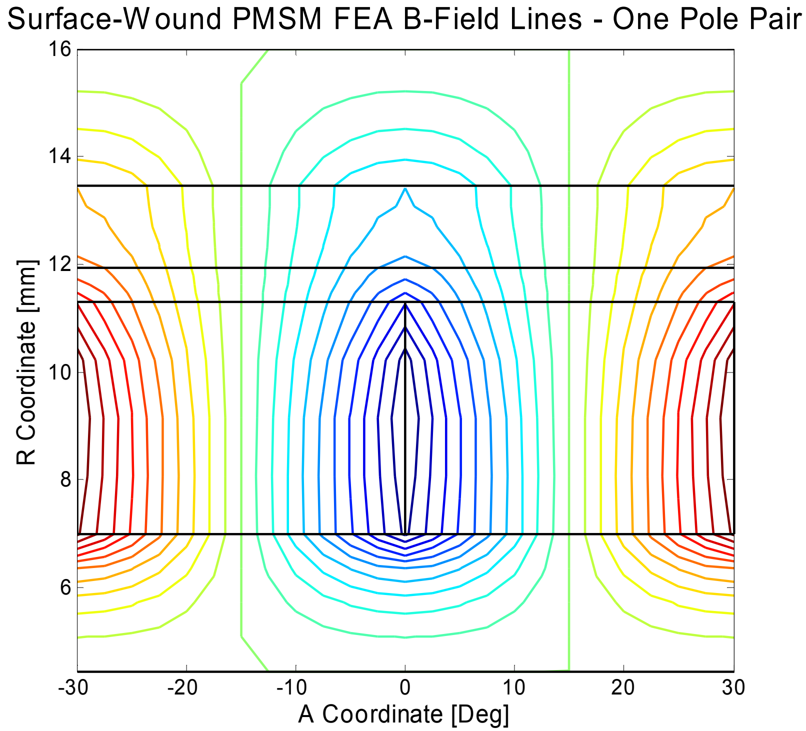
$$r = r_{ic} \qquad H_\theta\big|_{r=r_{ic}} = 0$$

Solution:

$$\varphi = K_0 r_{oc} \left[\frac{\left(\frac{r}{r_{ic}}\right)^s - \left(\frac{r_{ic}}{r}\right)^s}{\left(\frac{r_{oc}}{r_{ic}}\right)^s - \left(\frac{r_{ic}}{r_{oc}}\right)^s} \right] \sin(s\theta)$$

Verify Fields with FEA





Torque Laplace Solution = 8.3mNm

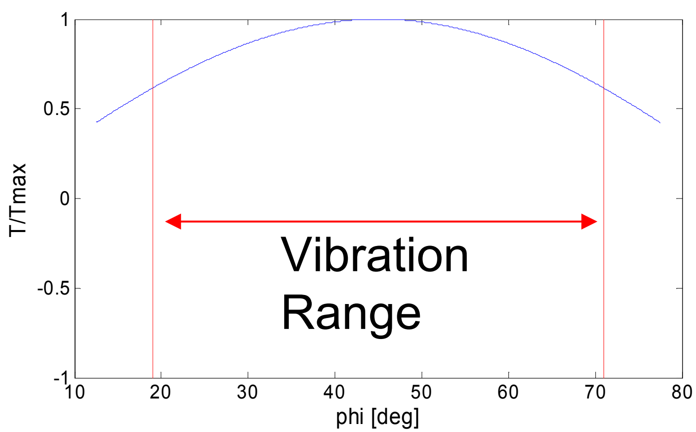
Torque FEA = 8.1mNm

Torque

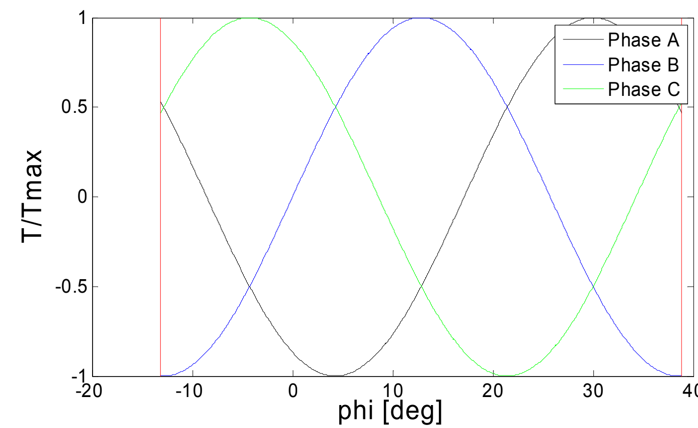
$$
\begin{aligned}
T_i &= r_m \int_{\theta=0}^{2\pi} \mu_0 M\big|_{r=r_m} H_\theta\big|_{r=r_m} z r_m d\theta \\
&= 4 r_m z \mu_0 M_0 K_0 s r_{oc} \left[\frac{\left(\frac{r_m}{r_{ic}}\right)^s - \left(\frac{r_{ic}}{r_m}\right)^s}{\left(\frac{r_{oc}}{r_{ic}}\right)^s - \left(\frac{r_{ic}}{r_{oc}}\right)^s} \right] \sin\left[s\left(\phi + \frac{2\pi(i-2)}{3} \right) \right]
\end{aligned}
$$

- s must equal p for efficient generation
- Φ = should maximize the sin function (Φ=90deg as often as possible)

Single phase



3 phase



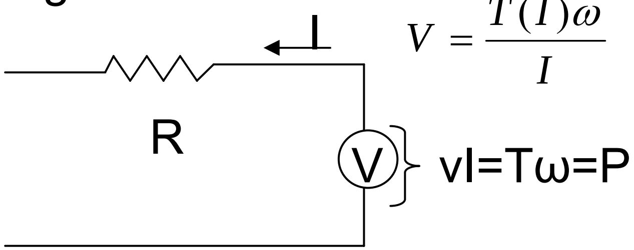
s should be chosen based on the number of phases in a single phase machine it is best to operate near the peak of the torque curve, but in a multi-phase machine s should allow operation over a full pull so as not to leave a weak phase which wastes potential current carrying material

Phases and Poles

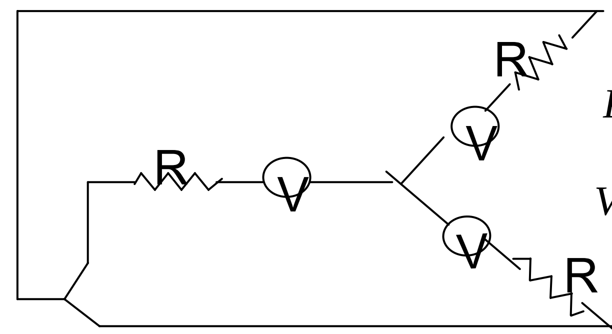
Single Phase

$$P \approx \frac{V^2}{4R}$$

$$V = \frac{T(I)\omega}{I}$$



Three Phase



$$P \approx \sum \frac{V^2}{4R}$$

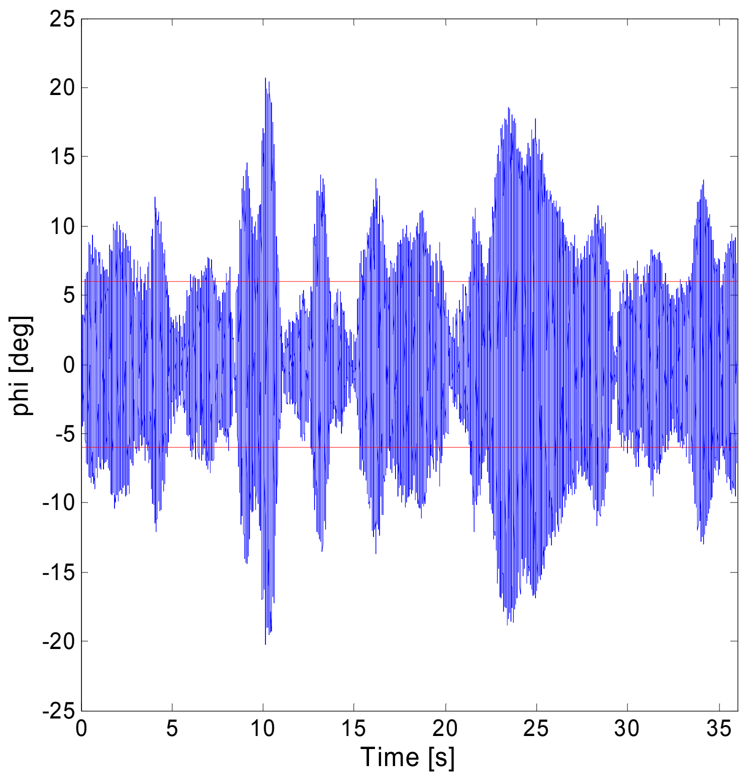
$$V = \frac{T(I)\omega}{I}$$

- To select the number of phases and the number of poles, use a passive control model (resistor), T = bω, and estimate the response of the system
- From the estimated response determine the appropriate number of poles based on the expected displacement as a function of the number of phases (operating over at least a pole pitch for a multi-phase machine or operating near the peak in a single-phase machine)
- Estimate the resistance in the coil based on the area and number of phases
- Calculate the voltage and subsequently the approximate power output as a function of the number of phases

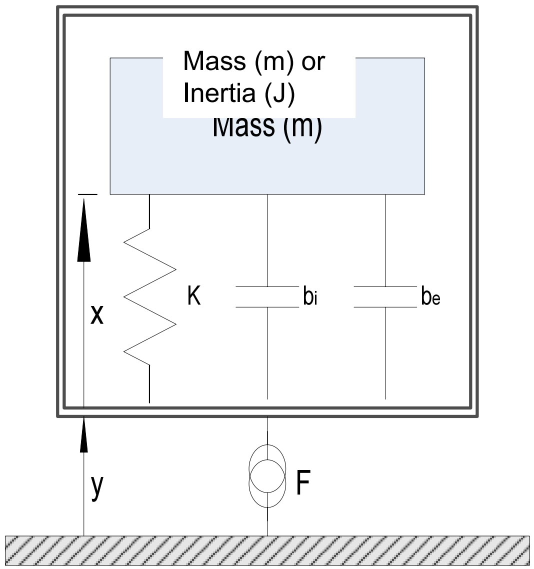
Number of Phases

- s single phase = 8
  - Magnet width ~0.125”
- s multi-phase = 30
  - Magnet width~0.05”
- This immediately suggests that a single phase system is better for this limited displacement application.

Single representative acceleration Trace used with the output power ($P=b\Phi^2$) optimized as a function of K and b



Models



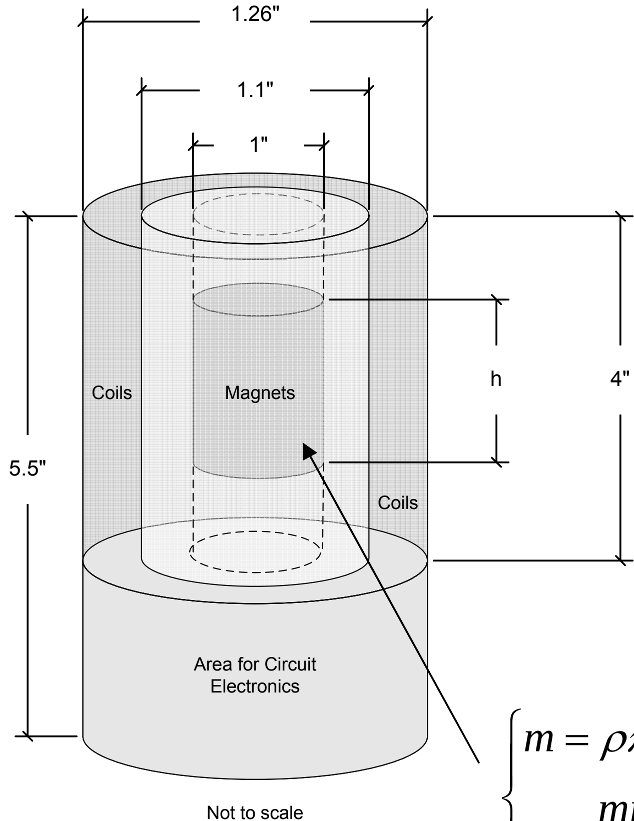
$$\ddot{x} + \frac{\left(b_i + b_e\right)_x}{m}\dot{x} + \frac{K_x}{m}x = -\ddot{y}$$

$$P = (b_e)_x \dot{x}^2$$

$$\ddot{\phi} + \frac{\left(b_i + b_e\right)_\phi}{J}\dot{\phi} + \frac{K_\phi}{J}\phi = -\ddot{\theta}$$

$$P = (b_e)_\phi \dot{\phi}^2$$

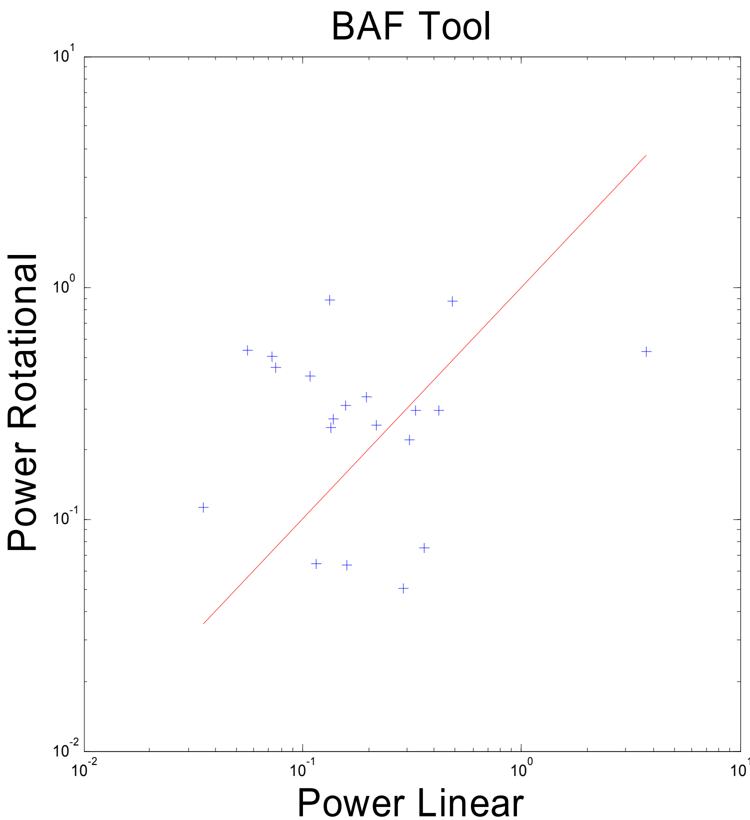
Parameters

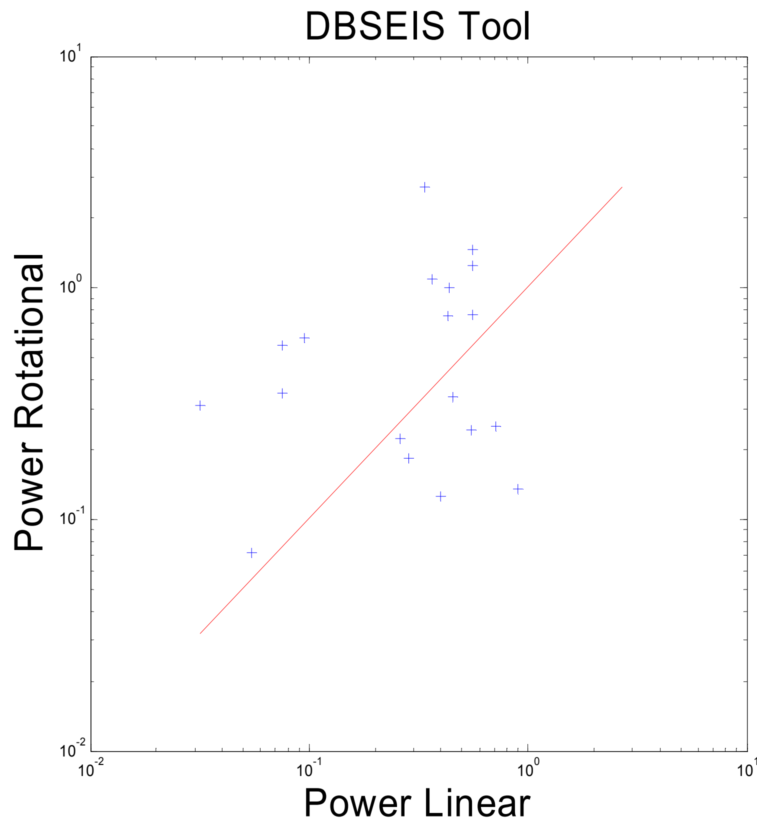


$$\begin{Bmatrix} m = \rho \pi r^2 h \\ J = \dfrac{m r^2}{2} \end{Bmatrix}$$

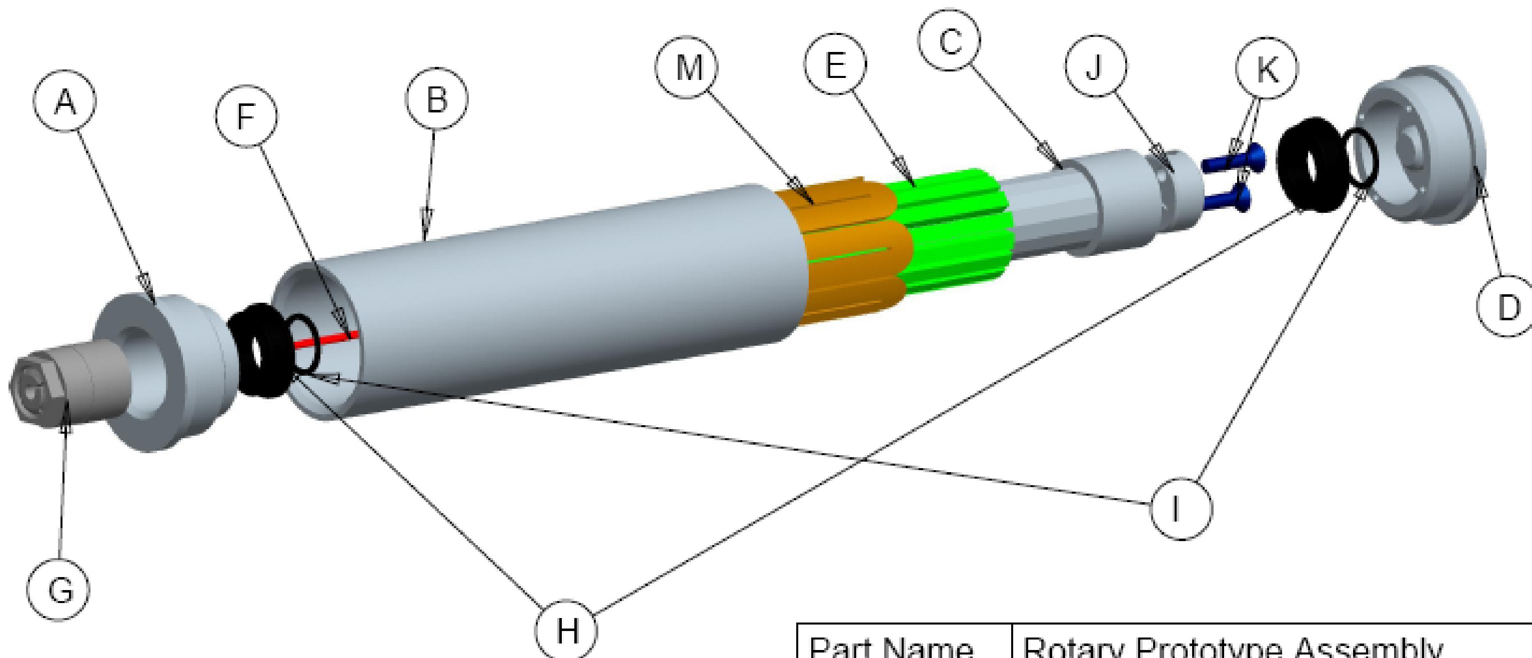
- Density/Inertia of moving mass is combination of steel core and magnets
  - ρ(steel)~7.8g/cc
  - ρ(magnet)~7.4g/cc
    - www.kjmagnetics.com

Comparison over all traces





Ref	Part	Qty	Notes
A	Upper Cap	1	Custom Manufacture
B	Casing	1	Custom Manufacture
C	Steel Core	1	Custom Manufacture
D	Lower Cap	1	Custom Manufacture
E	Magnet	48	www.magnet4less.com # NB030
F	Torsion Rod	1	www.timesavers.com
G	TranTorque MINI	1	McMaster Part # 5926K12
H	Bearing	2	17mm X 9mm McMaster 7804K148
I	Spring Washer	2	McMaster Part # 9714K26
J	Split Collar	1	McMaster Part #9961K11
K	4-40 screws	2	McMaster Part # 91253A110
L	Split Sleeve	2	
M	Winding	1	



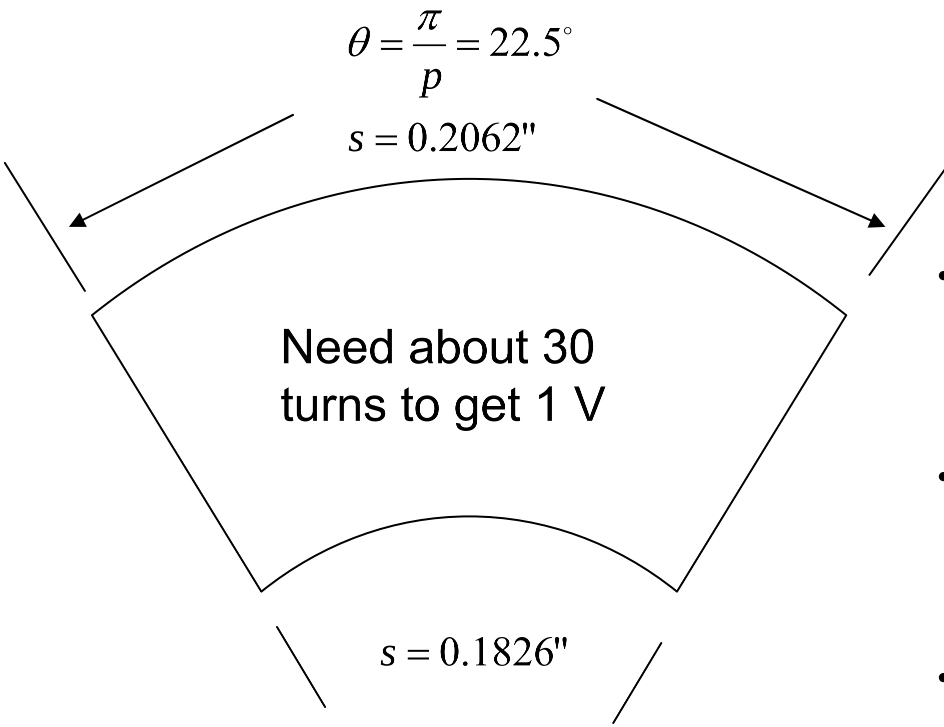
Part Name	Rotary Prototype Assembly
Contact	Zac Trimble atrimble@mit.edu 801-547-7795


1

2

3

4



- EDM copper
  - Radial Kerf = 0.004”
  - Tangential Kerf = 0.0055”
- Tech-Etch
  - Radial Kerf = 0.004”
  - Tangential Kerf = 0.002”
- Magnet wire
  - Insulation Thickness ~0.002”

Compaction Factor = Copper Area/Total Area (Total Area = 0.0117 in^2)

EDM = 0.61 Predicted Coil Resistance = 11 ohm

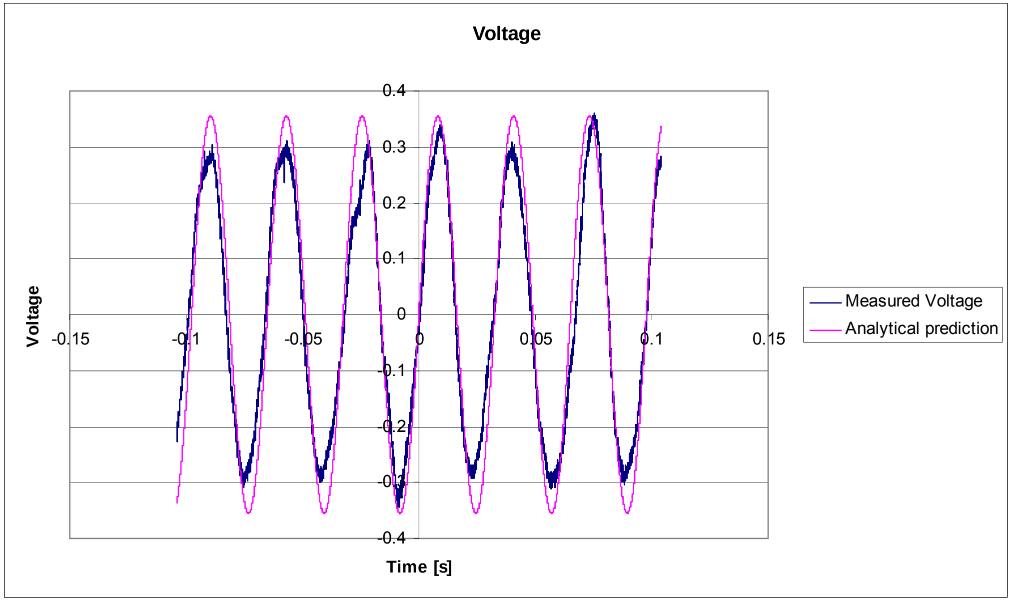
Tech = 0.78 Predicted Coil Resistance = 6 ohm

Wire = 0.66 Predicted Coil Resistance = 10 ohm (Measured 18 ohm)

Hand wound coil

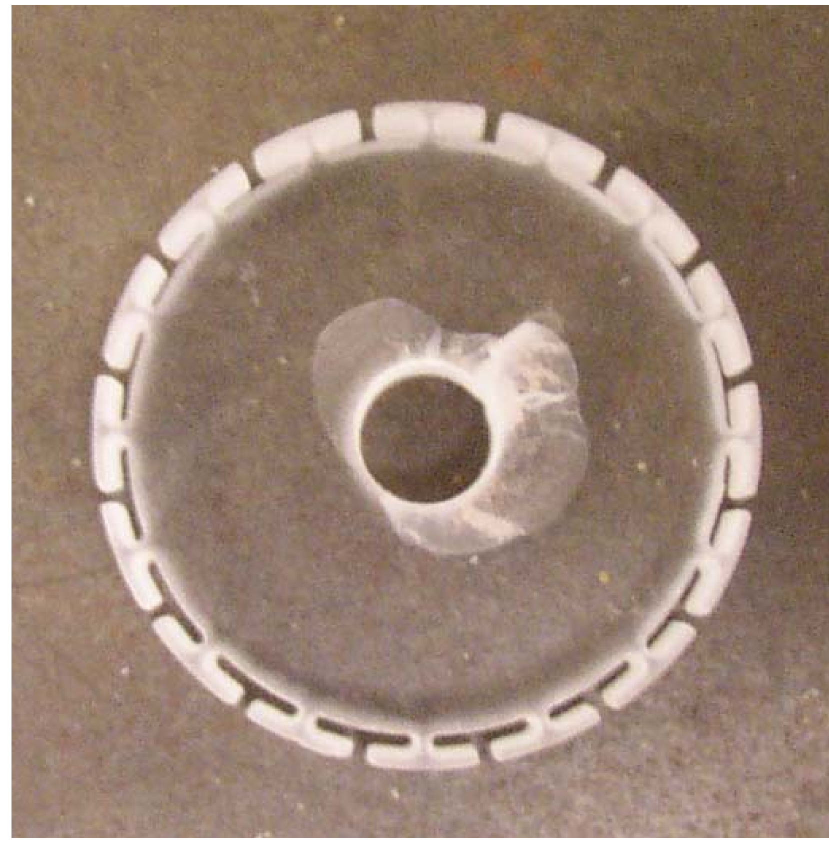
Prototype Performance

- Predicted voltage is 11% different than predicted.
    - Most likely cause is the winding manufacture
    - Other possible causes to explore are end effects and eddy current losses.



Water Jet laminations

- Water jet cut laminations as winding pattern
- Outer diameter of slots is 1.020 inner diameter of stator core is 1.050
- Inner diameter of slots 0.965 outer diameter of rotor is 0.913
- Inner hole is 0.22



Ream center holes

- Ream holes to 0.251 for assembly on mandrel

Slot end laminations

- Slot the end laminations to contain the end turns so the end turns are contained and don't get hit in post processing

Laminations Glued together Wires hand wound

- Laminations are glued together on a mandrel to maintain concentricity.
  - Unwound mass 48.08g
- Wires are then hand wound on the laminations
  - 20 turns
  - 14 ohm
  - Wound mass 56.59g
  - Compaction factor = 22.4%

Potting

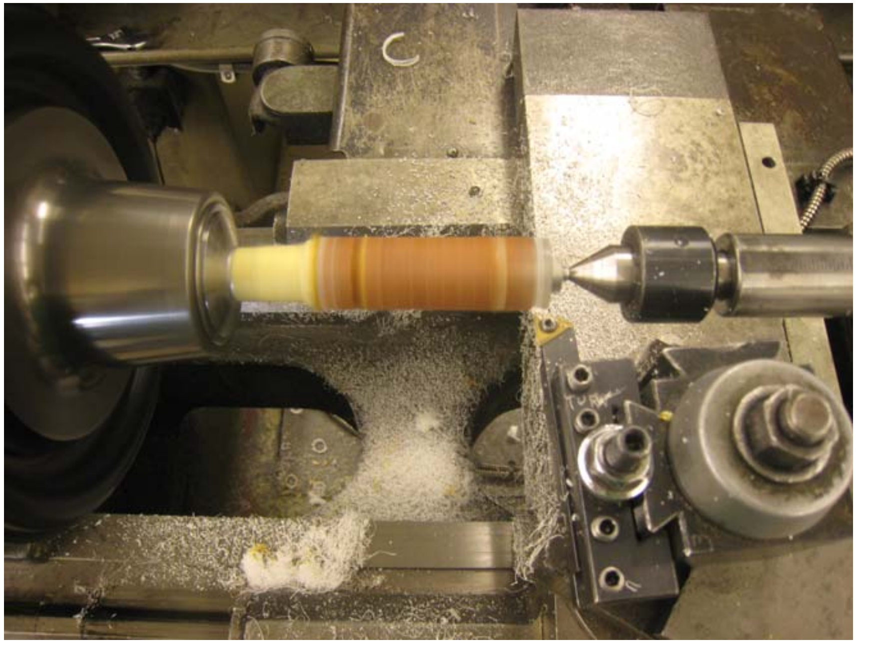
- Coil is placed into a pvc “mold” and vacuum potted in epoxy.

Coil post potting

- pvc split by hack saw and sperated.

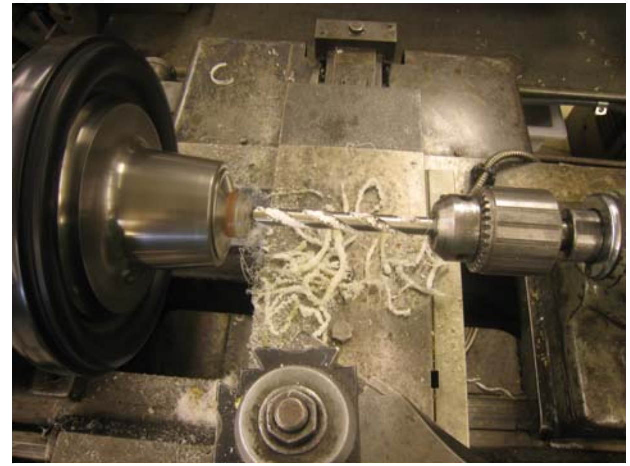
Turn outside to correct diameter

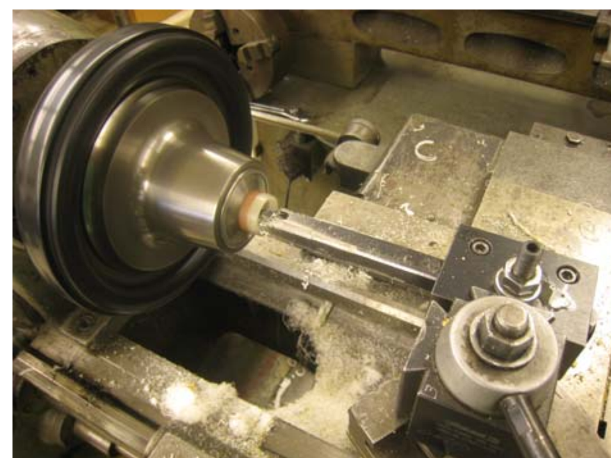
- Turn outside to 1.048 to fit the inside stator.



Drill and bore inner diameter

- Drill and bore inner diameter to 0.950





Existing Prototype

Existing Prototype

Vary Frequency

$$\omega_n = \sqrt{\frac{K}{m}},\ K \propto h^3$$

$\rightarrow$ 10's Hz

$\rightarrow$ vary h

$\rightarrow$ <10 Hz

$\rightarrow$ vary m

Increase Power

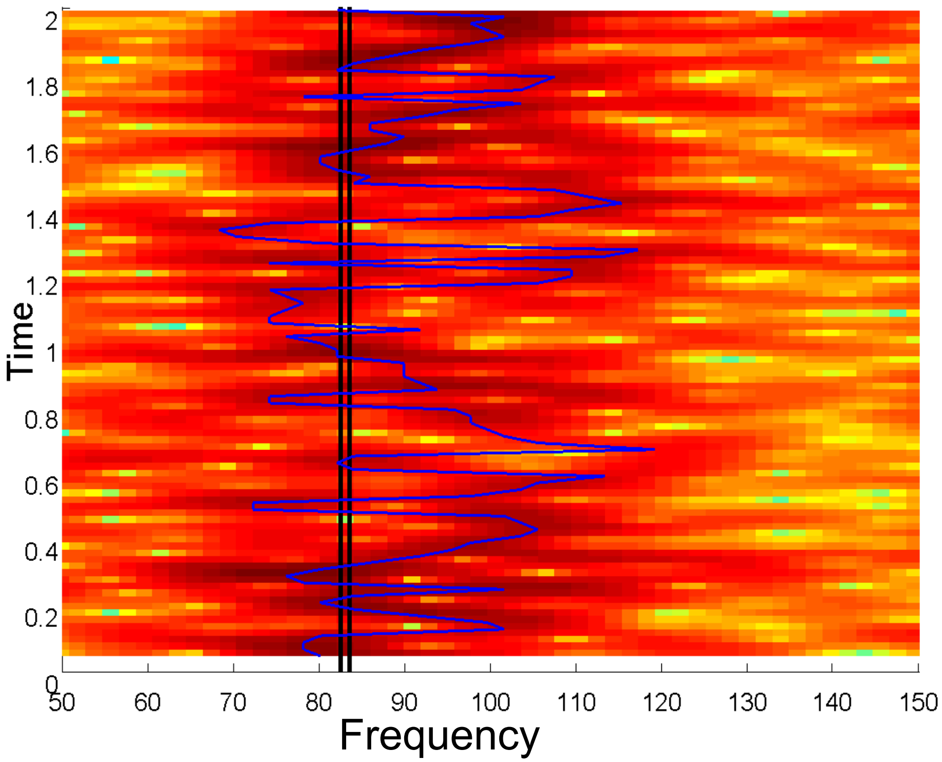
$$P_{avg\ max} = \frac{A_n^2 m Q}{4\omega_n}$$

$\rightarrow$ increase m

$\rightarrow$ increase Q

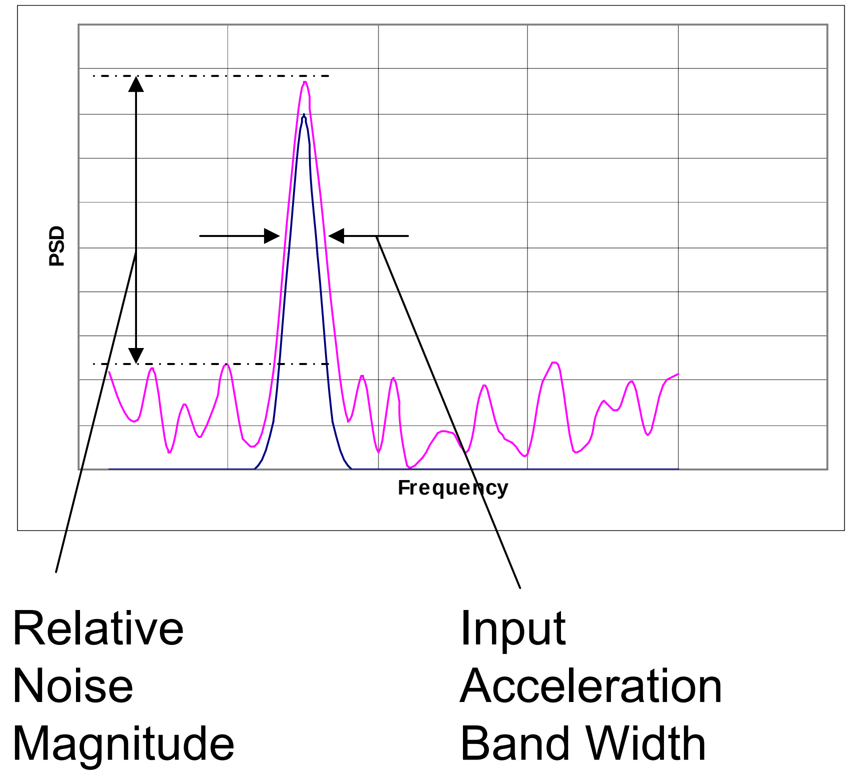
Floating Frequency Peak

- Spectrogram
  - Black lines = Device half power band width
  - Blue Line = Maximum amplitude
- Non-stable frequency peak



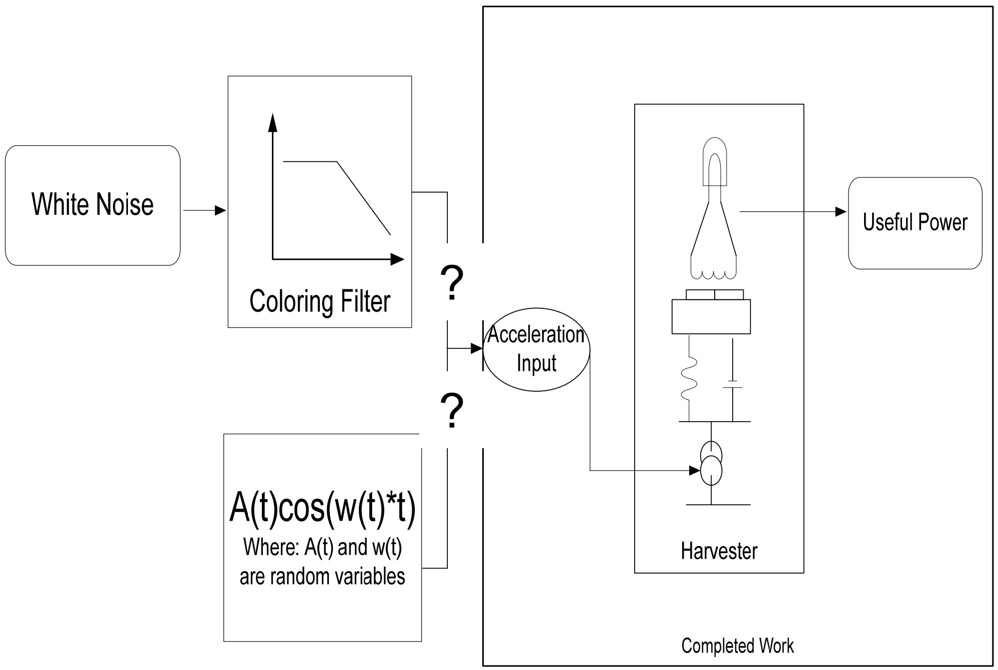
Current Work

- Lyapunov transformation model
  - stochastic, statistical state-space model
    - characterize expected response as a function of the band-width of the input, and the relative magnitude of the surrounding noise.
- Current Harvester used to verify numerical results



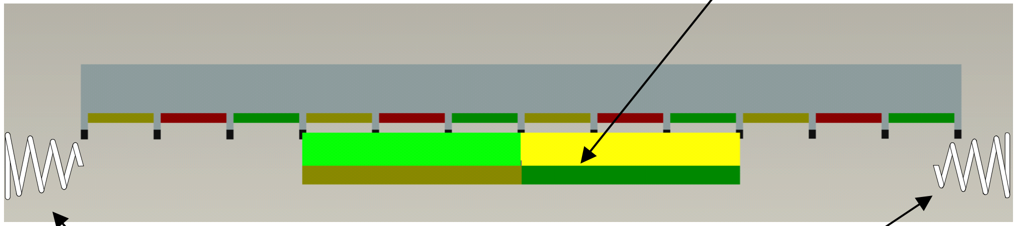
Current Work

- Modeling and testing of input signal
  - Modeling of input signal to determine the input characteristics, thus identifying which challenge is most restrictive (possibly reword)
    - Is the signal inherently wide band?
    - Is the signal narrow band with a unsteady phase?
    - Is the underlying signal steady with an on/off mechanical noise?
- Testing
  - Comparison of manufactured signal to measured acceleration data



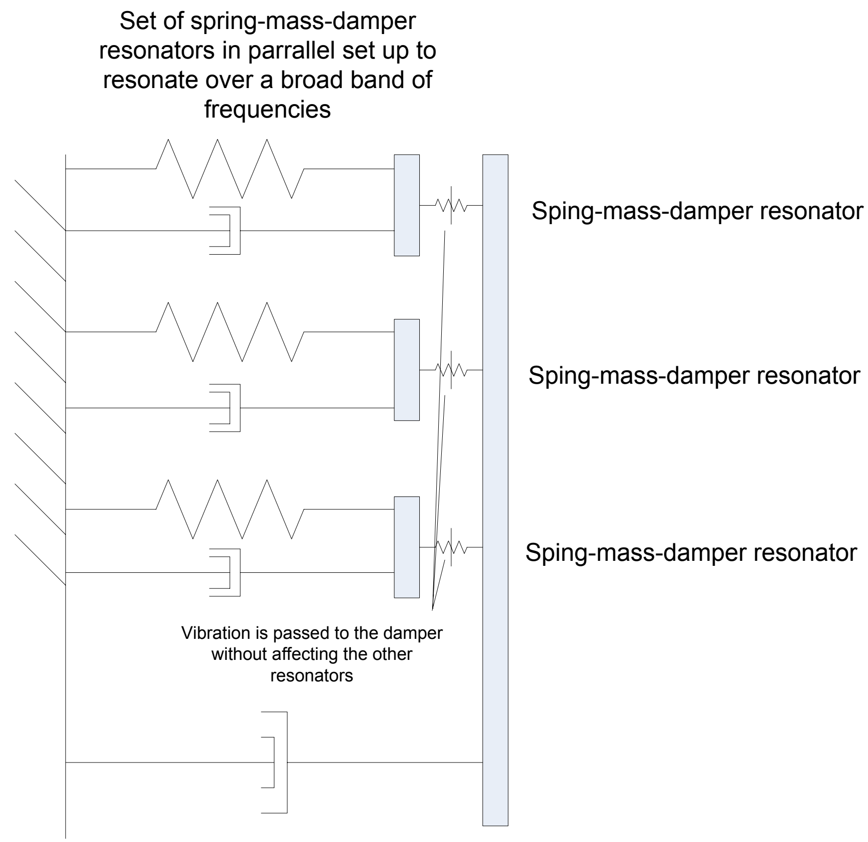
“Slug” Damper

- Eliminate Spring
  – Allow mass to move in “free space”, but constrain to near elastic collisions at displacement limits



“Piano Key” Fingers

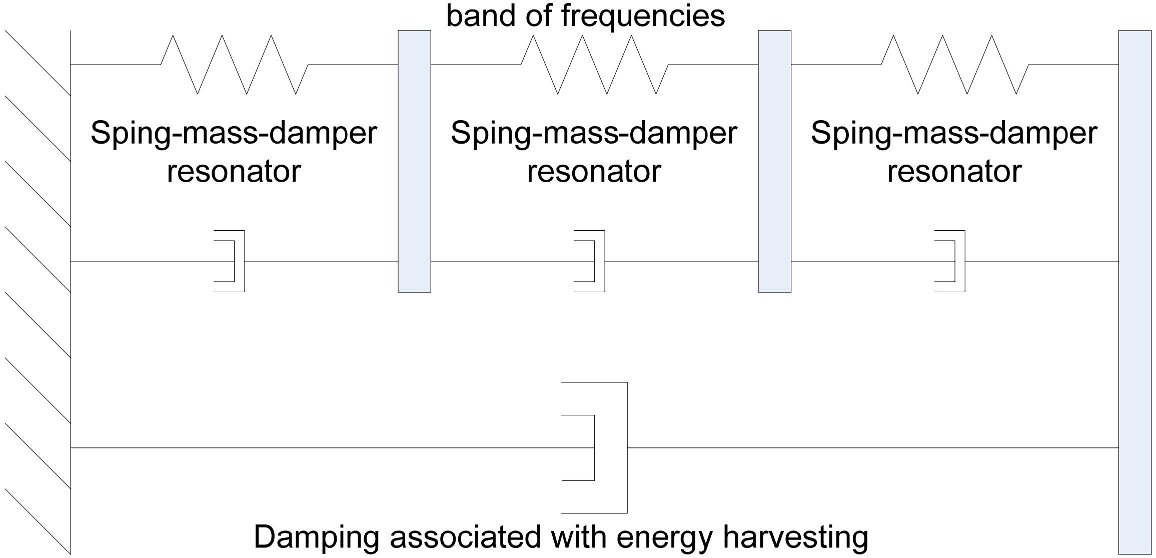
- Multiple resonators each tuned to a different frequency in the design range so that the entire bandwidth is covered
    - Research questions
        - How many resonators
        - Can each resonator be isolated



Higher Order System

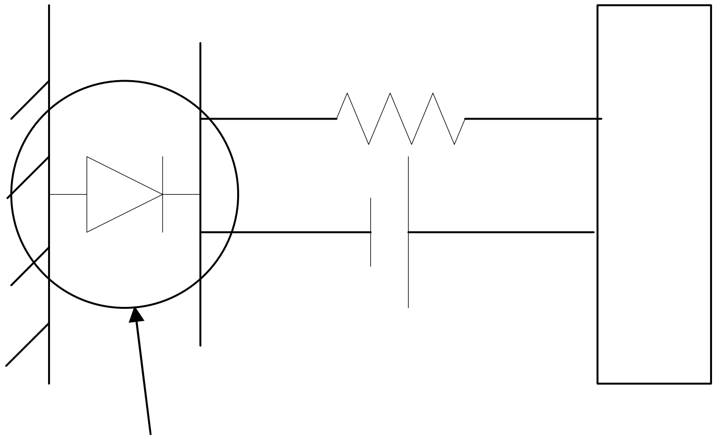
Schematically understood as a series arrangement of resonators whose governing differential equation is tunable to a larger band-width

Set of spring-mass-damper resonators in series set up to resonate over a broad band of frequencies



Mechanical Rectifier

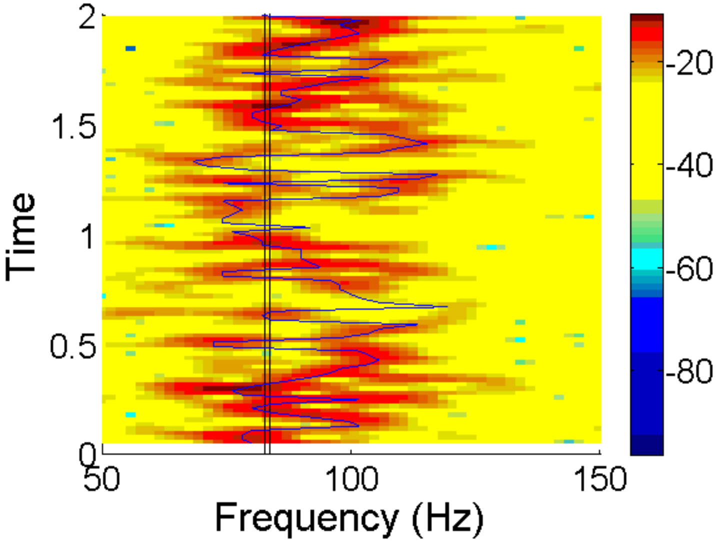
- Connect the harvester to the vibrating environment through a mechanical rectifier that passes a periodic signal
  - Band pass filter
  - Periodic Impulse filter



Mechanical Rectifier

Frequency Tracking

- Change the harvester frequency by changing the effective spring constant to follow a variable frequency input.



$$\omega_n(t) = \sqrt{\frac{K(t)}{m}}$$

Energy Harvesting Update 5/28/2009

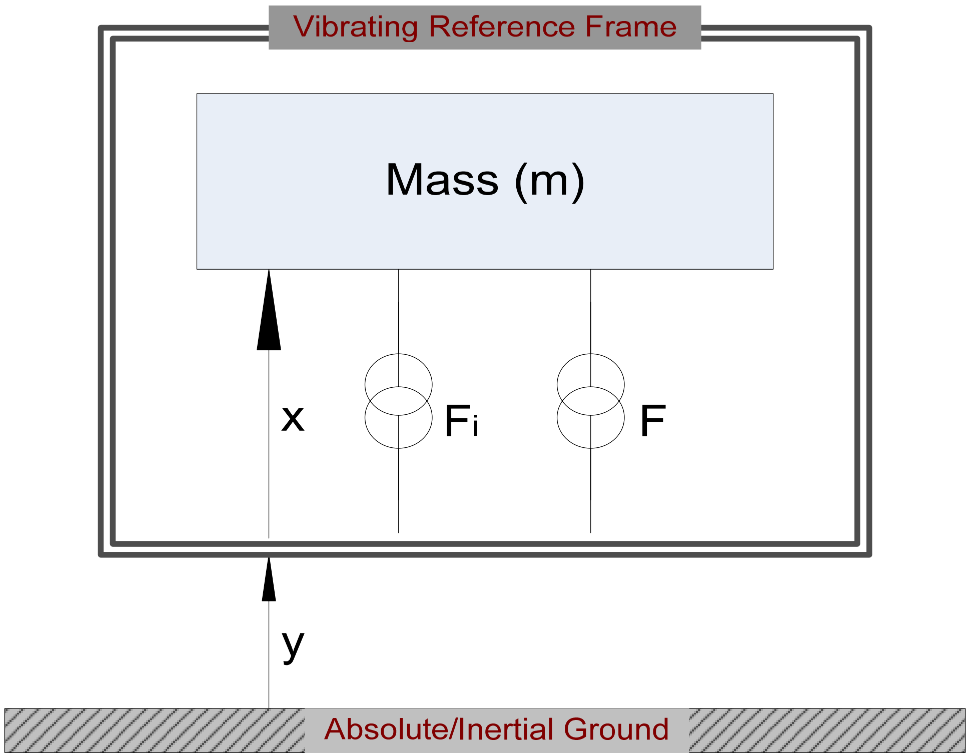
Zac Trimble	MIT Ph.D. candidate
Jahir Pabon	SDR
Alex Slocum	MIT Professor – Mechanical Engineering
Jeff Lang	MIT Professor – Electrical Engineering Computer Science

Future Work

- Controls
  - Apply optimal control solutions to optimize the performance over a wider bandwidth.
- Mechanical tracking
- Ratchets/Clutches.
  - One of the advantages of a rotational system is the potential for greater

Problem Formulation

- General Problem Statement:
  - Given a known acceleration input to a vibrating reference frame, determine the maximum amount of power that can be extracted from the given vibration.



- Problem assumptions/specifics
  - To extract power, a proof mass is assumed to be attached to the reference frame by a force F
  - An additional force, F_i, associated with unavoidable internal losses also connects the proof mass to the reference frame.

- Using an optimal control approach, determine the force that will extract the most power from the relative motion between the reference frame and proof mass.

Optimization Solution

Since the maximum is a limiting case of *F*, find *F** by looking at the boundaries.

$$
\begin{aligned}
H(v^*, p^*, F^*, t) &\geq H(v^*, p^*, F, t) \\
\left(v^* - \frac{p^*}{m}\right) F^* - p^* \left(\frac{F_i}{m} + a\right) &\geq \left(v^* - \frac{p^*}{m}\right) F - p^* \left(\frac{F_i}{m} + a\right) \\
\left(v^* - \frac{p^*}{m}\right) F^* &\geq \left(v^* - \frac{p^*}{m}\right) F
\end{aligned}
$$

Thus, the optimal control is a "bang-bang" control where the force is always set to the maximum possible in the direction defined by $\textsc{Signum}\left[v^* - \frac{p^*}{m}\right]$

$$
F^* = F_{max}\ \textsc{Signum}\left[v^* - \frac{p^*}{m}\right]
$$

Controls Summary

- Additional optimization schemes that involve models of the signal.
- For new data check if signal “tracking” can be incorporated

Ratchet/Clutch

- Design is modular and is set up to incorporate a clutch or ratchet to force continuous rotation
  - Eliminate reversal points
  - Maximize average velocity
- Develop ratchet design and model

Mechanical tracking

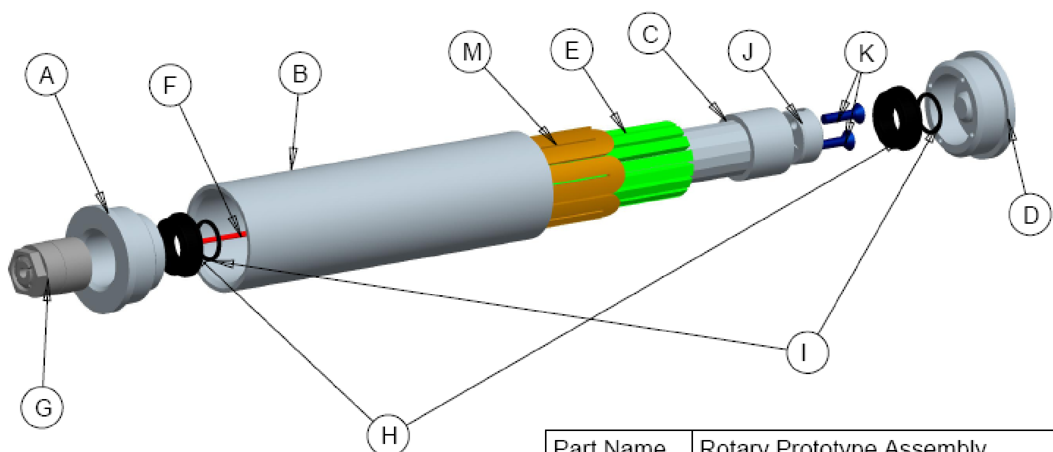
- Finish modeling of possible mechanical tracking
  – Actively changing stiffness by adjusting spring variables
  – Passively change stiffness by incorporating stiffening/relaxing springs

Additional Future Work

- Additional prototype testing
  - Rotational shaker
- New “Gyro” data
- Refine power prediction model

Questions

Ref	Part	Qty	Notes
A	Upper Cap	1	Custom Manufacture
B	Casing	1	Custom Manufacture
C	Steel Core	1	Custom Manufacture
D	Lower Cap	1	Custom Manufacture
E	Magnet	48	www.magnet4less.com # NB030
F	Torsion Rod	1	www.timesavers.com
G	TranTorque MINI	1	McMaster Part # 5926K12
H	Bearing	2	17mm X 9mm McMaster 7804K148
I	Spring Washer	2	McMaster Part # 9714K26
J	Split Collar	1	McMaster Part #9961K11
K	4-40 screws	2	McMaster Part # 91253A110
L	Split Sleeve	2	
M	Winding	1	

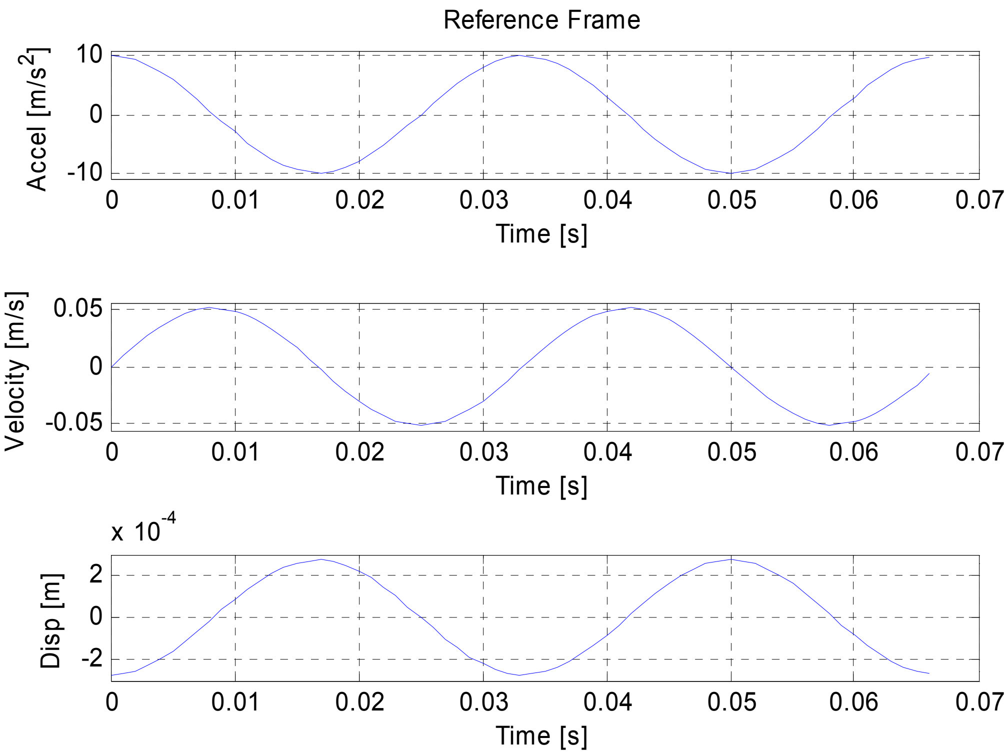


Part Name	Rotary Prototype Assembly
Contact	Zac Trimble atrimble@mit.edu 801-547-7795

Optimal Control

1/27/2009

Harmonic Reference Input



$$\ddot{y}(t) = \mathrm{Re}\left[Ae^{j\omega t}\right]$$

$$\dot{y}(t) = \mathrm{Re}\left[\frac{A}{j\omega}e^{j\omega t}\right]$$

$$y(t) = \mathrm{Re}\left[\frac{A}{(j\omega)^2}e^{j\omega t}\right]$$

A = 1g

ω= 30Hz

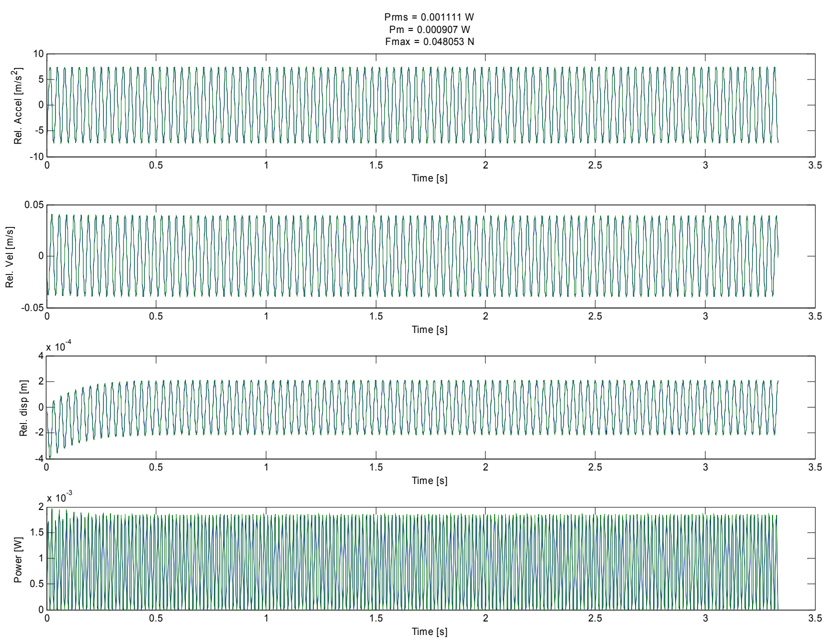
Viscous Damping Baseline

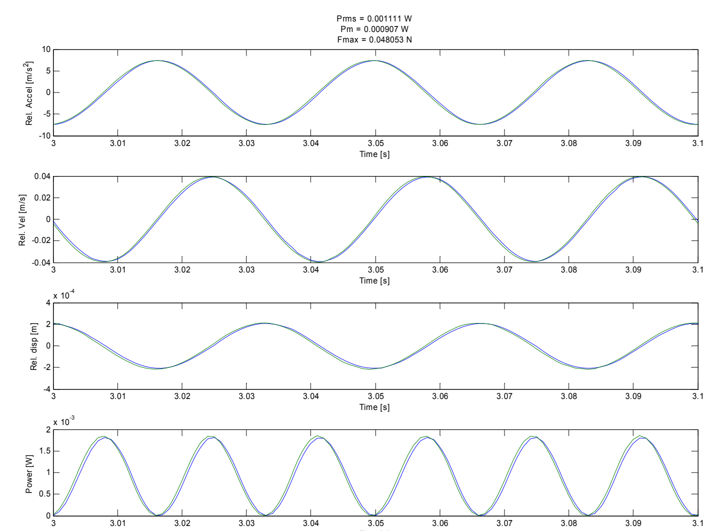
$$P(t) = F(t)\dot{x}(t) = b\dot{x}^2(t)$$

b for max power output = 1.191 N-s/m

Fmax = 0.04805 N

Pmean= 0.9 mW



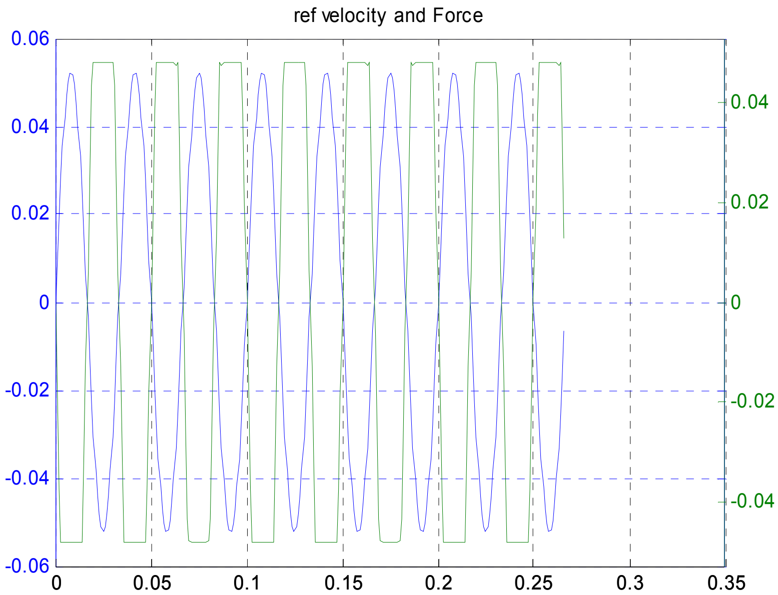


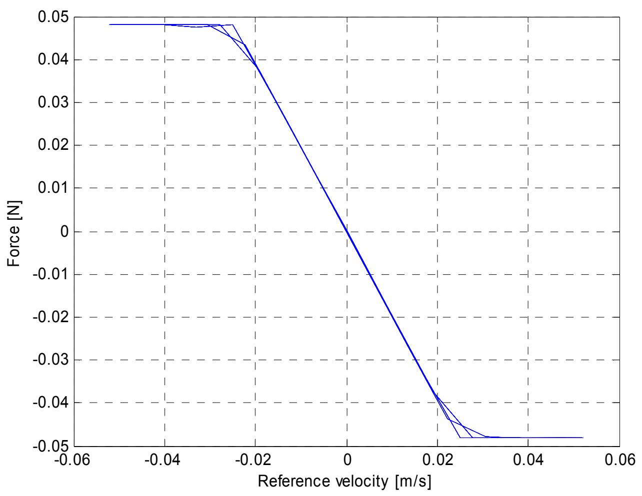
Numerical Force

Numerically determining the optimal force no assumptions other than abs(F)≤Fmax where Fmax is the same as before.

Note: due to solution method, only 8 cycles are counted (sampling frequency 1000Hz)

Pmean = 1.5 mW =167% of viscous damper





Viscous Damper Plus Spring

Resonant spring system:

w = 30Hz

wn =29.99Hz

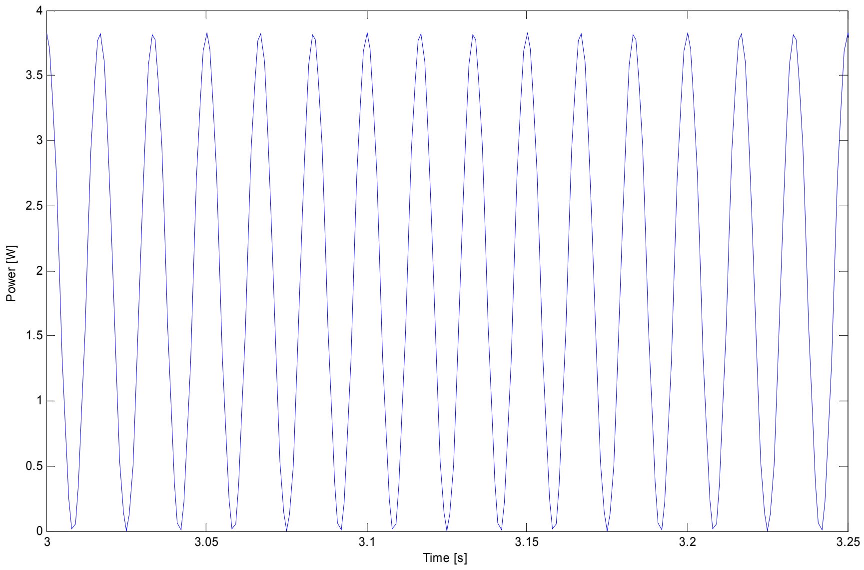
ζ = 0.01

b = 0.57

Fmax damper = 1.47 N

Fmax damper + spring = 73.58 N

Power = 1.92 W



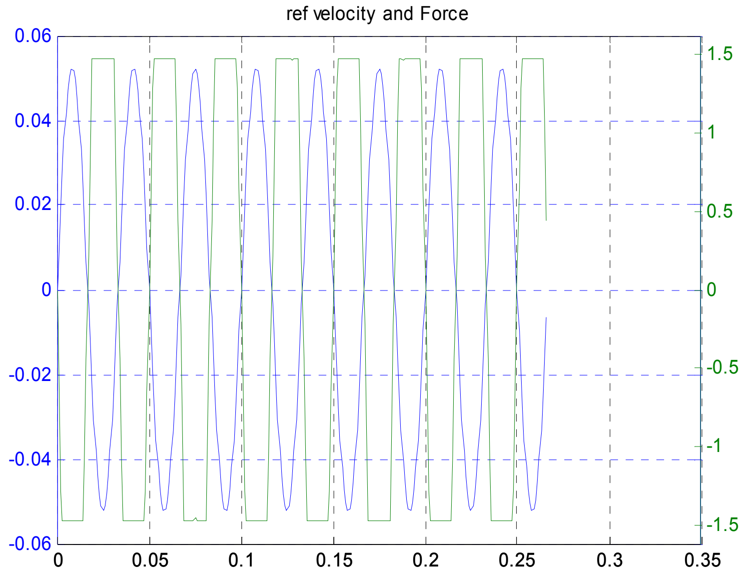
Numerical Simple plus spring

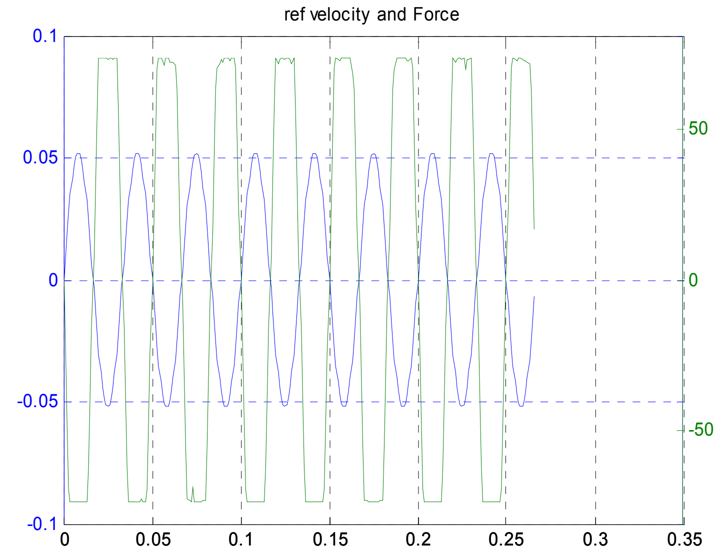
Fmax = 1.47 N

Power = 47.1 mW
= 2.4% of smd

Fmax = 73.58 N

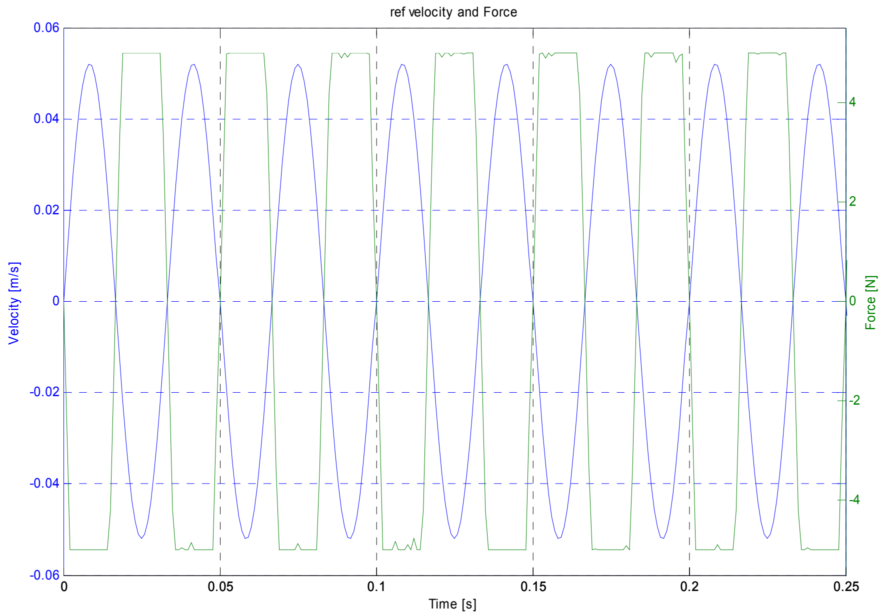
Power = 2.28 W
= 119% of smd

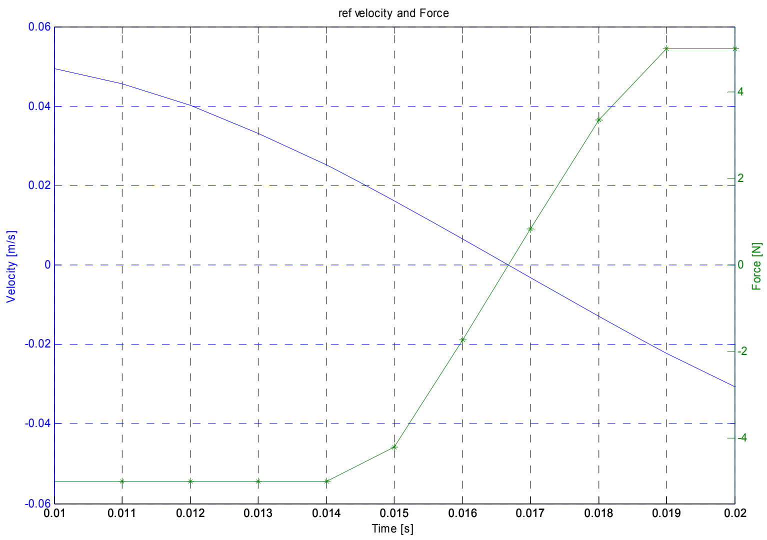




Numerical Continuous

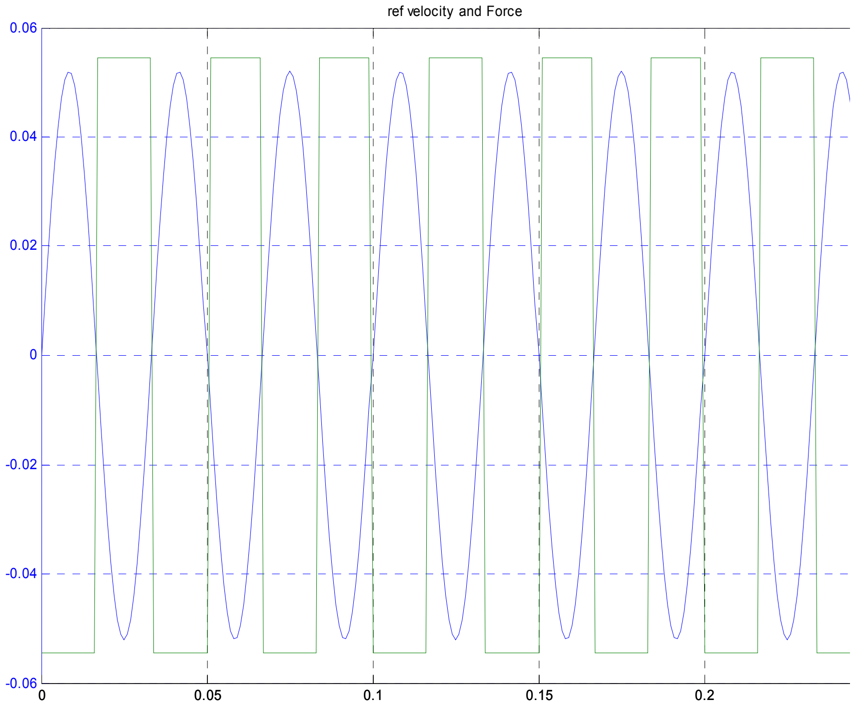
Fmax = 5 N;   Pavg = 160.8 mW

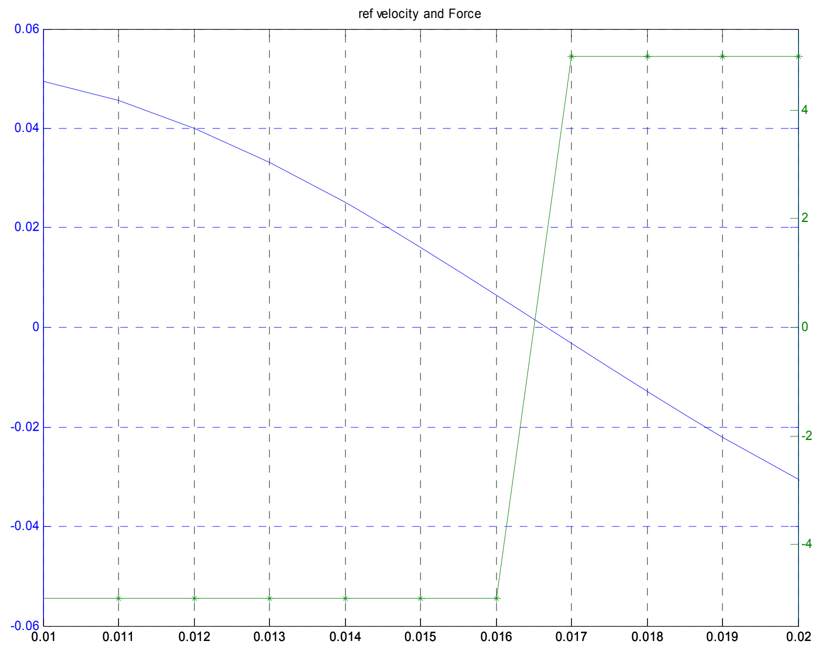




Numerical Forced Binary

Fmax = 5 N; Pavg = 158.7 mW

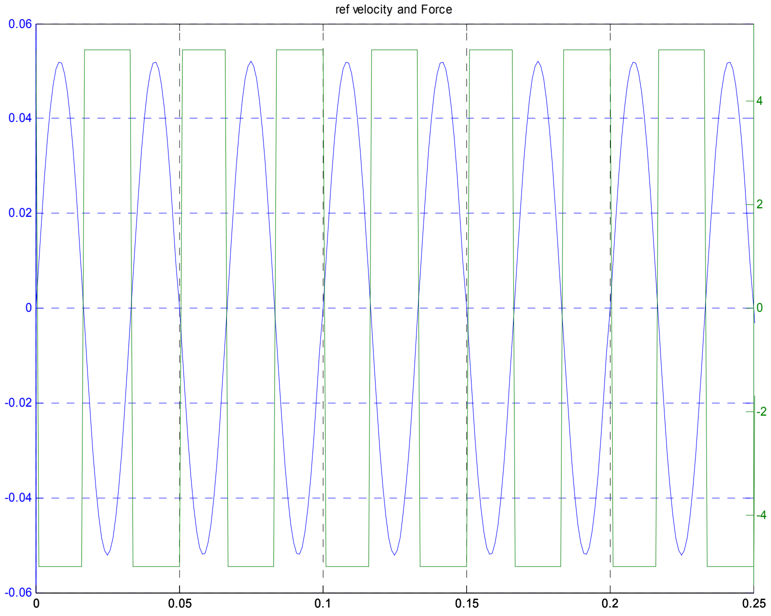


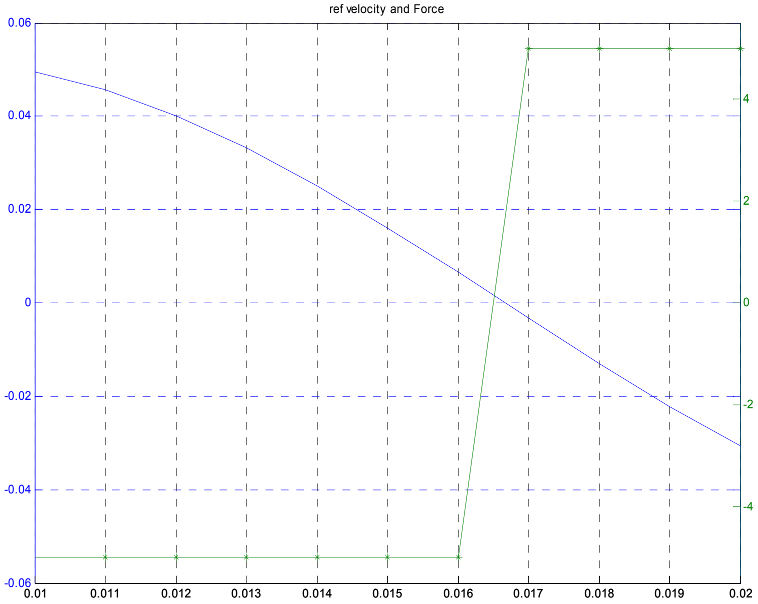


Numerical Forced Phase

Fmax = 5 N;   Pavg = 165 mW

Phase = pi;

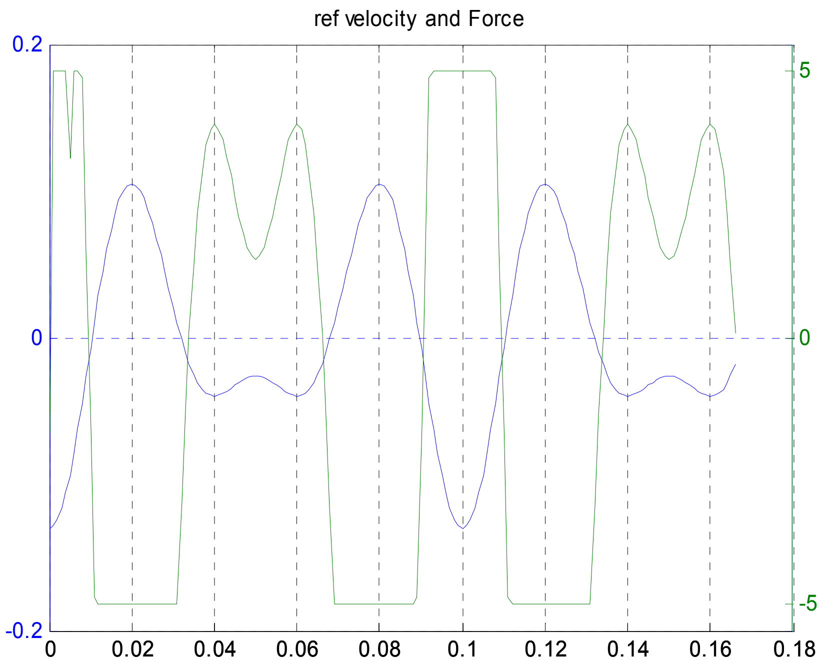


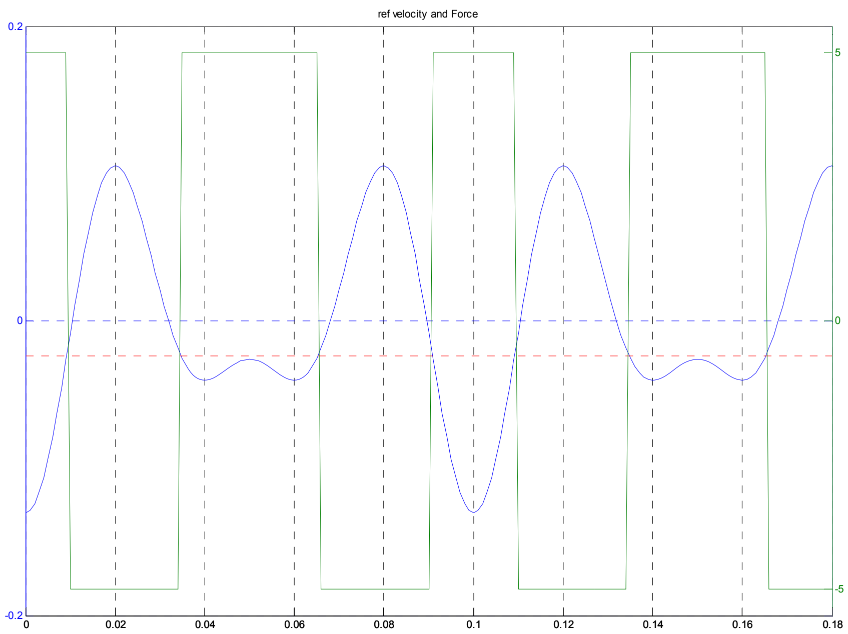


Numerical Forced 20,30 Hz sin

Fmax = 5 N;
Pavg = 262 mW;

Fmax = 5 N
Pavg = 281 W

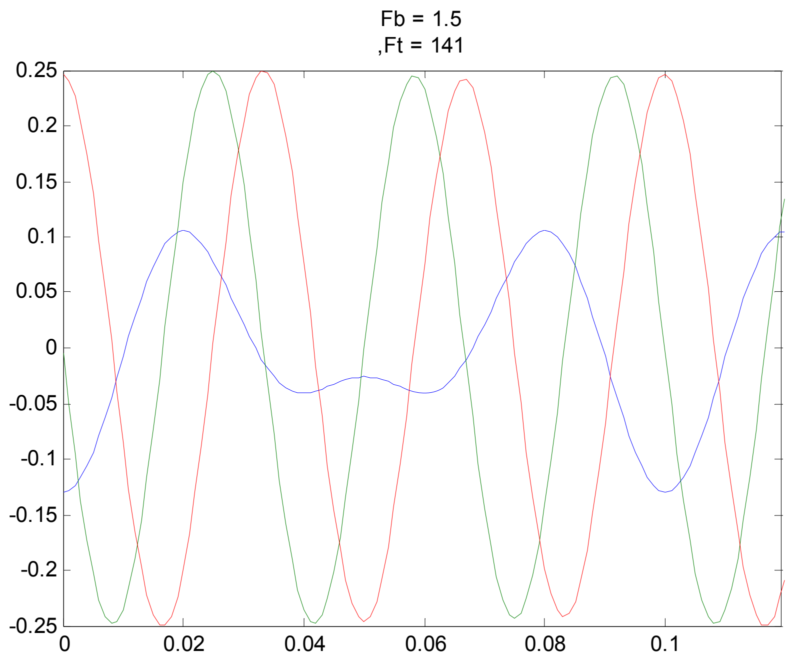


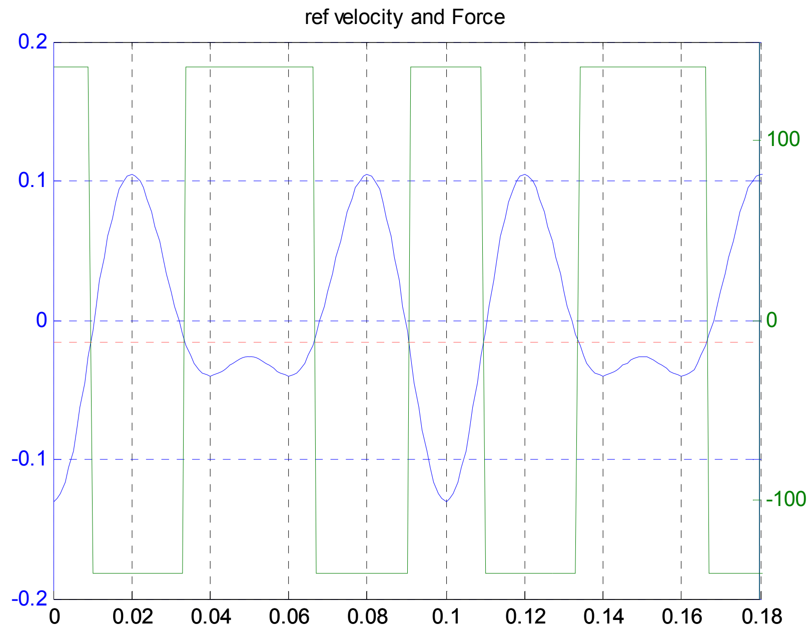


Numerical Forced 20,30 Hz sin

b = 0.3 N-s/m;
K = 5329 N/m;
Pavg = 3.16 W;

Pavgb = 94 mW;
Pavgt = 7 W;





Viscous Damper Plus Spring

Resonant spring system:

w = 30Hz

wn =29.99Hz

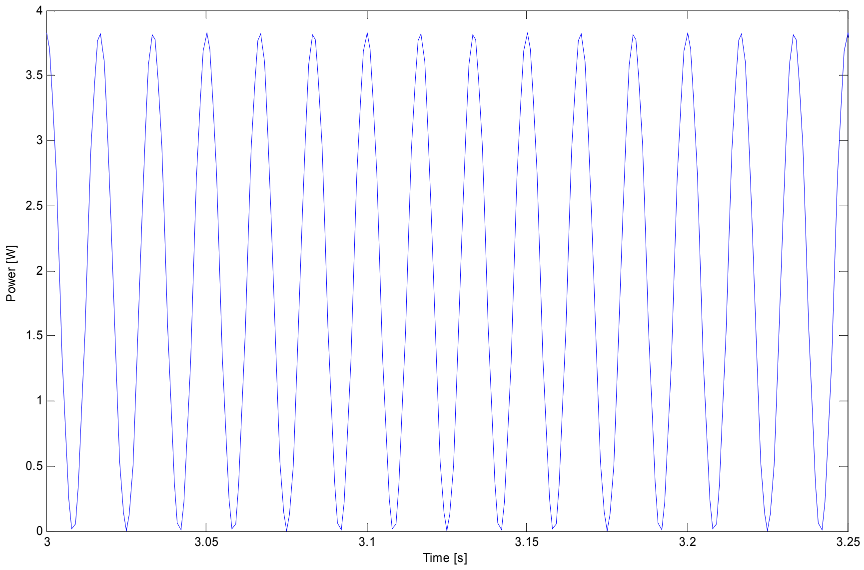
ζ = 0.01

b = 0.57

Fmax damper = 1.47 N

Fmax damper + spring = 73.58 N

Power = 1.92 W

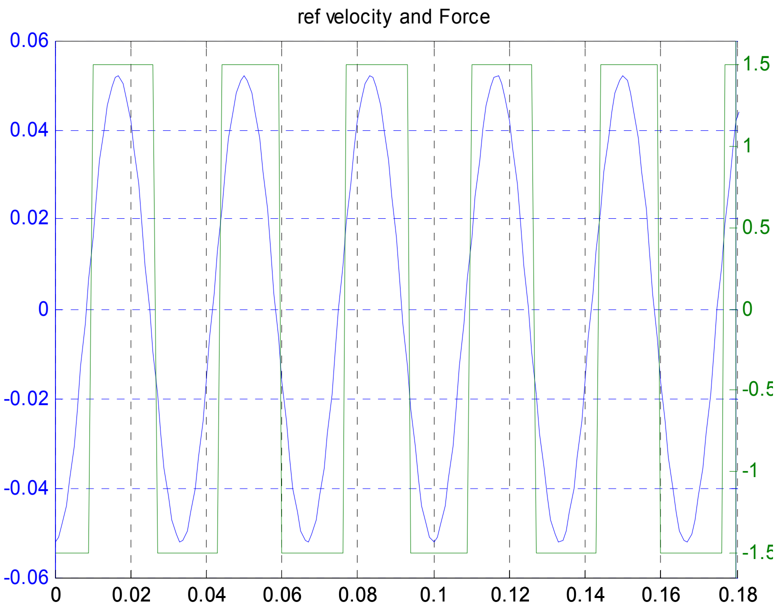


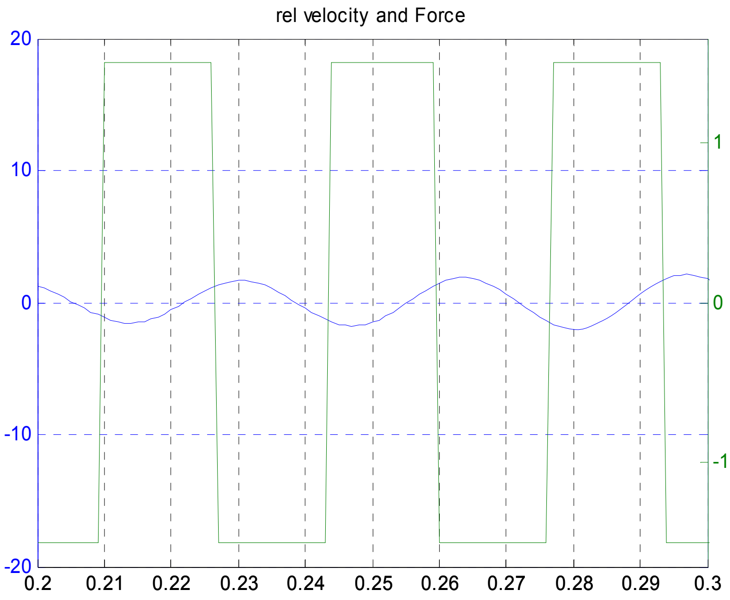
Spring + Optimal Force

Fmax = 1.5 N;

K = 5330 N/m;

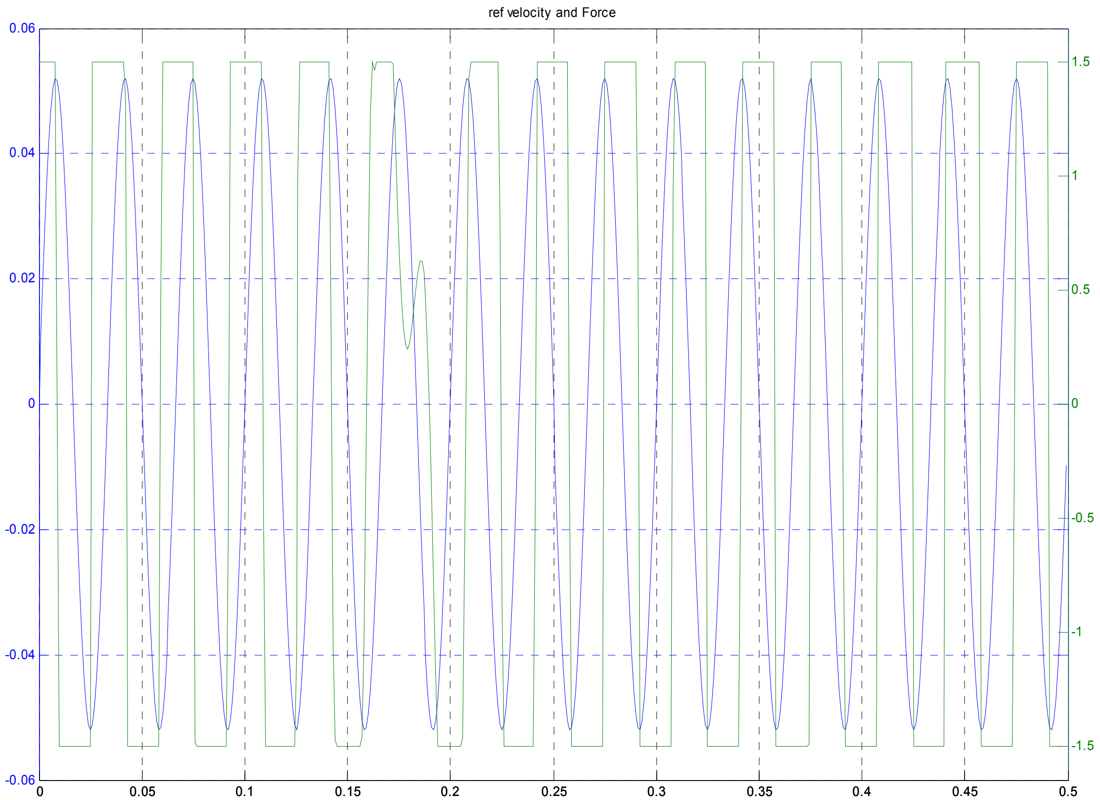
P = 3.87 W;





Straight numerical

- P = 797mW



Optimal Control Estimate

The following documents the process used to produce a best case estimate for power output from the provided acceleration inputs using an optimal control approach.

General Optimal Control Problem

- The optimal control problem seeks to maximize a performance functional, J, by controlling a trajectory, **x**, with control input **u**.

Governing Equation:	$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \quad \text{with IC} \quad \mathbf{x}(t_0) = \mathbf{x}_0$
Definition of Performance Functional:	$J(\mathbf{u}) = K[\mathbf{x}(t_1)] + \int_{t_0}^{t_1} L[\mathbf{x}(t), \mathbf{u}(t), t]dt$

- To find the value of **u** which maximizes J subject to the governing equation, define the Hamiltonian of the system. (Where **p** is the costate, and <> denotes inner product)

Hamiltonian:	$H(\mathbf{x}, \mathbf{p}, \mathbf{u}, t) = L(\mathbf{x}, \mathbf{u}, t) + \langle \mathbf{p}, \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \rangle$

- The optimal control input **u*** which will maximize J must then satisfy the following conditions.

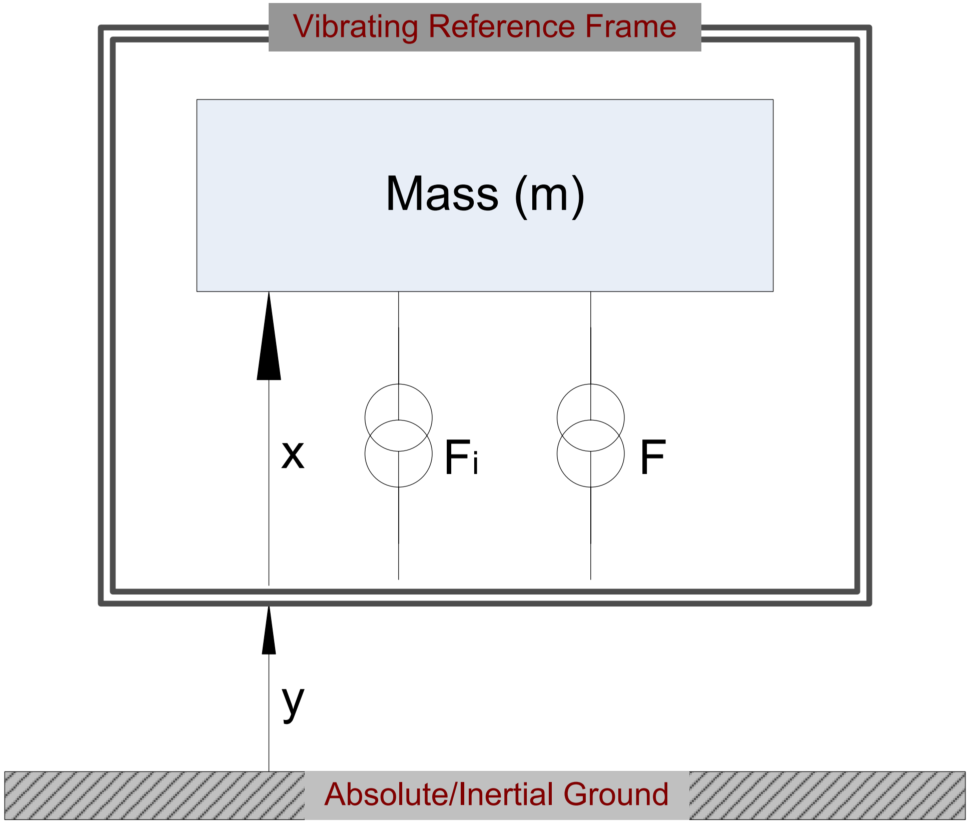
1.	$\dot{\mathbf{x}}^*(t) = \frac{\partial H}{\partial \mathbf{p}}, \; \mathbf{x}^*(t_0) = \mathbf{x}_0$ $\dot{\mathbf{p}}^*(t) = -\frac{\partial H}{\partial \mathbf{x}}, \; \mathbf{p}^*(t_1) = \frac{\partial K}{\partial \mathbf{x}}$

2.	$\left.\frac{\partial H}{\partial \mathbf{u}}\right\|_* = 0$
3.	$\left.\frac{\partial^2 H}{\partial \mathbf{u}^2}\right\|_*$ is negative definite

Problem Formulation

•General Problem Statement:

•Given a known acceleration input to a vibrating reference frame, determine the maximum amount of power that can be extracted from the given vibration.



•Problem assumptions/specifics

•To extract power, a proof mass is assumed to be attached to the reference frame by a force F

•An additional force, F_i, associated with unavoidable internal losses also connects the proof mass to the reference frame.

•Using an optimal control approach, determine the force that will extract the most power from the relative motion between the reference frame and proof mass.

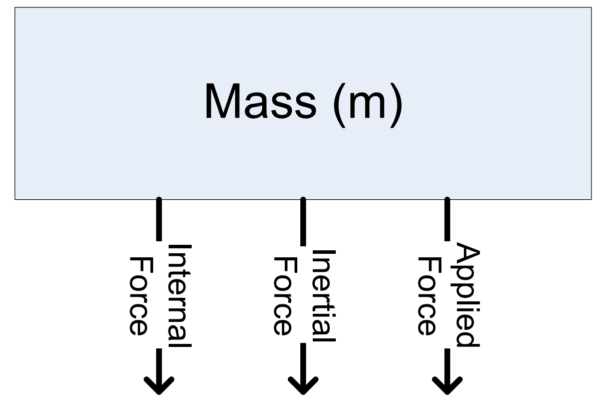
This formulation looses generality by assuming a zero impedance source

Formulation of Governing Equation

•Summation of the forces on a free body diagram of the proof mass, provides the governing equation of motion for the proof mass.

$$
\begin{aligned}
0 &= F_{inertial} + F_{internal} + F \\
0 &= m\frac{d^2}{dt^2}(x+y) + F_i + F \\
\frac{d^2x}{dt^2} &= -\frac{1}{m}(F_i + F) - \frac{d^2y}{dt^2} \\
(\ddot{x}) &= -\frac{1}{m}(F_i + F) - \ddot{y} \\
\dot{v} &= -\frac{1}{m}(F_i + F) - a
\end{aligned}
$$

Where, $\begin{cases} a \equiv \ddot{y} \\ v \equiv \dot{x} \end{cases}$



FBD

•Define the state variable of the system to be the relative velocity between the proof mass and reference frame, v

•Define the control variable as the applied force, *F*

Definition of the Performance Functional and subsequent Hamiltonian

- The goal of any energy harvester is to extract the maximum amount of power from the environment possible. Thus, the performance functional for the system is the total power extracted over a given time span from 0→T

$$Power_{avg} = \frac{1}{T}\int_0^T Fv dt = \int_0^T \frac{Fv}{T} dt$$

- Thus,

$$K\left[v(T)\right] \equiv 0 \qquad \text{and,} \qquad L\left[v(t), F(t), t\right] \equiv \frac{Fv}{T}$$

- and,

$$H(v, p, F, t) = \left(\frac{v}{T} - \frac{p}{m}\right) F - p\left(\frac{F_i}{m} + a\right)$$

total power

Optimization Solution

-Condition 1

Equation 1

Equation 2

Note, dimensionally, the physical interpretation of the costate is force.

total power

Optimization Solution
-Condition 2

The chosen performance functional is linear with respect to *F*, thus, the second condition for optimality is independent of the input,

This suggests that the optimal costate is physically related to the relative momentum of the proof mass divided by time.

Since the second condition is independent of the control input, the control input that will maximize the performance functional must be a limit or boundary of the input. This changes the final condition to a straight inequality to determine which limit maximizes H.

Optimization Solution

-Condition 3

Since the maximum is a limiting case of *F*, find *F** by looking at the boundaries.

Thus, the optimal control is a “bang-bang” control where the force is always set to the maximum possible in the direction defined by

General Questions

- What would be an appropriate modification to the performance functional to ensure the prediction of a resonant solution?
  - A penalty on small x?

Definition of the Performance Functional and subsequent Hamiltonian total power

- The goal of any energy harvester is to extract the maximum amount of power from the environment possible. Thus, the performance functional for the system is the total power extracted over a given time span from 0→T

$$Power_{total} = \int_0^T Fv\,dt$$

- Thus,

$$K\left[v(T)\right] \equiv 0 \qquad \text{and,} \qquad L\left[v(t), F(t), t\right] \equiv Fv$$

- and,

$$H(v,p,F,t) = \left(v - \frac{p}{m}\right)F - p\left(\frac{F_i}{m} + a\right)$$

Question: Is the proper performance metric the average power or the total power?

total power

Optimization Solution

-Condition 1

Equation 1

$$\begin{aligned} \dot{v}^* &= \frac{\partial H}{\partial p}(v^*, p^*, F^*, t) \\ &= -\frac{1}{m}(F_i + F^*) - a \end{aligned}$$

Equation 2

$$\begin{aligned} \dot{p}^* &= -\frac{\partial H}{\partial v}(v^*, p^*, F^*, t) \\ &= -\left[F^* - \frac{p^*}{m}\frac{\partial F_i}{\partial v}\right] \\ &= \frac{p^*}{m}\frac{\partial F_i}{\partial v} - F^* \end{aligned}$$

$$\dot{p}^* - \frac{1}{m}\frac{\partial F_i}{\partial v}p^* = -F^*$$

Note, dimensionally, the physical interpretation of the costate is momentum.

Optimization Solution

-Condition 2

The chosen performance functional is linear with respect to *F*, thus, the second condition for optimality is independent of the input,

$$\left.\frac{\partial H}{\partial F}\right|_{*} = v^{*} - \frac{p^{*}}{m} = 0$$

$$mv^{*} = p^{*}$$

This suggests that the optimal costate is physically related to the relative momentum of the proof mass.

Since the second condition is independent of the control input, the control input that will maximize the performance functional must be a limit or boundary of the input. This changes the final condition to a straight inequality to determine which limit maximizes H.

Optimization Solution

-Condition 3

Since the maximum is a limiting case of *F*, find *F** by looking at the boundaries.

$$
\begin{aligned}
H(v^*, p^*, F^*, t) &\geq H(v^*, p^*, F, t) \\
\left(v^* - \frac{p^*}{m}\right) F^* - p^* \left(\frac{F_i}{m} + a\right) &\geq \left(v^* - \frac{p^*}{m}\right) F - p^* \left(\frac{F_i}{m} + a\right) \\
\left(v^* - \frac{p^*}{m}\right) F^* &\geq \left(v^* - \frac{p^*}{m}\right) F
\end{aligned}
$$

Thus, the optimal control is a "bang-bang" control where the force is always set to the maximum possible in the direction defined by $\textsc{Signum}\left[v^* - \frac{p^*}{m}\right]$

$$
F^* = F_{max}\ \textsc{Signum}\left[v^* - \frac{p^*}{m}\right]
$$